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# Sequence of Maps Between Hopf and Aharonov-Bohm Bundles

M. Socolovsky

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria, 04510, México D. F., México
Email: socolovs@nucleares.unam.mx

**Abstract** The existence of the Aharonov-Bohm ($A-B$) effect with its associated $U(1)$-bundle $\xi_{A-B}$, and the uniqueness -up to homotopy- of a continuous function $S^2 \to \mathbb{C}^*$, induce a unique map -up to isomorphism- between the Hopf bundles with zero and unit Chern number, respectively $\xi_0 : S^1 \to S^2 \times S^1 \to S^2$ and $\xi_1 : S^1 \to S^3 \to S^2$. This establishes a tight relation between $\xi_0$ and $\xi_1$ through $\xi_{A-B}$, and therefore between the $A-B$ effect and the hypothetical unit magnetic charge when the Dirac connection in $\xi_1$ is considered.

**Keywords:** Aharonov-Bohm effect, Hopf bundles, magnetic charge

## 1 Introduction

In a recent paper [1] we showed that, if $\xi_{A-B} : S^1 \to (T_0^2)^* \xrightarrow{pr_1} (D_0^2)^*$ and $\xi_D \equiv \xi_1 : S^1 \to S^3 \xrightarrow{\pi_D} S^2$ are the $U(1)$-bundles where the Aharonov-Bohm ($A-B$) [2] effect and the unit Dirac monopole ($D$) [3,4] are represented, then $\xi_{A-B}$ turns out to be isomorphic to the pull-back of $\xi_D$ induced by the inclusion of the corresponding base spaces: $\iota : \mathbb{C}^* \to S^2$. ($S^1 = U(1)$, $S^2 \cong \mathbb{C} \cup \{\infty\}$, $(D_0^2)^* \cong \mathbb{R}^{2^*} \cong \mathbb{C}^*$ is the punctured open 2-disk, $(T_0^2)^* = (D_0^2)^* \times S^1$ is the open solid 2-torus minus a circle, $S^k$, $k$=1,2,3, are spheres, $pr_1$ is the projection in the first entry, and $\pi_D$ is the Hopf map $\pi_D(z_1, z_2) = z_1/z_2$ if $z_2 \neq 0$ and $\infty$ if $z_2 = 0$, with $||(z_1, z_2)|| = 1$.) In particular, in the context of bundle theory, this allows to prove through the pull-back of the Dirac connection and the use of the Dirac quantization condition, the well known fact that the $A-B$ effect vanishes when the flux in the solenoid equals an integer times the quantum of magnetic flux $2\pi\hbar c/|e|$ associated with the electric charge $|e|$. Thus, one obtains a consequence on the observed $A-B$ effect [5,6] from the up to now only hypothetical existence of magnetic charges i.e. a sort of "passage" from $D$ to $A-B$.

An immediate question that arises is if it is possible an "inverse passage": from $A-B$ to $D$, which might, at least from a purely mathematical point of view, reinforce the idea of the real existence of magnetic charges. In this note, we try to come close to a positive answer to this question using the fact that, up to homotopy, the *unique* continuous map from the base space of $\xi_D$ to the base space of $\xi_{A-B}$ is a constant map:

$$\kappa : S^2 \to \mathbb{C}^*, \ b \mapsto \kappa(b) = z_0 \tag{1}$$

where $z_0$ is an arbitrary (non-zero) complex number.

In section **2.** we show that the pull-back of $\xi_{A-B}$ induced by $\kappa$ (or any other function in the homotopy class of $\kappa$ [7]) is isomorphic to the trivial Hopf bundle $\xi_0 : S^1 \to S^2 \times S^1 \xrightarrow{pr_1} S^2$ corresponding to zero Chern number, and therefore to zero magnetic charge in the bundle theory of magnetic monopoles. In section **3.**, through the composition of bundle maps, we obtain a canonical map $\xi_0 \to \xi_1$, thus establishing a tight relation between $\xi_0$, the "sandwiched" bundle $\xi_{A-B}$, and $\xi_1$, where the unit magnetic charge is described. Section **4.** is devoted to a conclusion.

## 2 Isomorphism Between $\kappa^*(\xi_{A-B})$ and $\xi_0$

$\kappa$ induces the pull-back bundle $\kappa^*(\xi_{A-B})$ with total space $P_{\kappa^*(\xi_{A-B})} \subset S^2 \times (\mathbb{C}^* \times S^1)$ defined by

$$P_{\kappa^*(\xi_{A-B})} = \{(b, (z, e^{i\varphi}))|\kappa \circ pr_1 = pr_1 \circ pr_2\} \tag{2}$$

 

i.e. such that the following diagram commutes:

$$\begin{array}{ccc}
P_{\kappa^*(\xi_{A-B})} \times S^1 & \stackrel{pr_2 \times Id_{S^1}}{\longrightarrow} & (\mathbb{C}^* \times S^1) \times S^1 \\
\psi^*_{A-B} \downarrow & & \downarrow \psi_{A-B} \\
P_{\kappa^*(\xi_{A-B})} & \stackrel{pr_2}{\longrightarrow} & \mathbb{C}^* \times S^1 \\
pr_1 \downarrow & & \downarrow pr_1 \\
S^2 & \stackrel{\kappa}{\longrightarrow} & \mathbb{C}^*
\end{array}$$

*Diagram 1*

where $\psi_{A-B}$ and $\psi^*_{A-B}$ are the right actions of $S^1$ over $P_{\xi_{A-B}} \cong \mathbb{C}^* \times S^1$ and $P_{\kappa^*(\xi_{A-B})}$, and $pr_2$ is the projection in the second entry. In fact, for the lower part of the diagram one has $\kappa \circ pr_1(b, (z, e^{i\varphi}))\kappa(b) = z_0 = pr_1 \circ pr_2(b, (z, e^{i\varphi})) = pr_1(z, e^{i\varphi}) = z$, and so

$$P_{\kappa^*(\xi_{A-B})} = \{(b, (z_0, e^{i\varphi})),\ b \in S^2,\ \varphi \in [0, 2\pi)\} \tag{3}$$

while for the upper part of the diagram it holds $\psi_{A-B} \circ (pr_2 \times Id_{S^1}) = pr_2 \circ \psi^*_{A-B}$:
$\psi_{A-B} \circ (pr_2 \times Id_{S^1})((b, (z, e^{i\varphi})), e^{i\lambda}) = \psi_{A-B}(pr_2((b, (z, e^{i\varphi})), e^{i\lambda}) = \psi_{A-B}((z, e^{i\varphi}), e^{i\lambda}) = (z, e^{i(\varphi+\lambda)})$,
$pr_2 \circ \psi_{A-B}((b, (z, e^{i\varphi})), e^{i\lambda}) = pr_2(b, (z, e^{i(\varphi+\lambda)})) = (z, e^{i(\varphi+\lambda)})$.

Since $z_0$ is a fixed (but otherwise arbitrary) non-zero complex number,

$$\varPhi : P_{\kappa^*(\xi_{A-B})} \to S^2 \times S^1,\ (b, (z_0, e^{i\varphi})) \mapsto (b, e^{i\varphi}) \tag{4}$$

is an homeomorphism, and therefore one has the bundle isomorphism

$$\xi_0 \stackrel{(Id_{S^2}, \varPhi^{-1})}{\longrightarrow} \kappa^*(\xi_{A-B}) \tag{5}$$

at the extreme left of Diagram 2.

*Remark*: The existence of an isomorphism $\xi_0 \to \kappa^*(\xi_{A-B})$ can be proved from the fact that $\iota \circ \kappa$ is constant and therefore the Chern class of $\kappa^*(\xi_{A-B})$ is zero.

## 3 Bundle Map $\xi_0 \to \xi_D$

Putting together the pull-back of $\xi_D$ by $\iota$, namely $\iota^*(\xi_D)$, and the isomorphism $\xi_{A-B} \stackrel{(Id_{\mathbb{C}^*}, \varPsi^{-1})}{\longrightarrow} \iota^*(\xi_D)$, respectively given in sections 4. and 5. of Ref. [1], and the results of the previous section, one obtains the sequence of bundle maps

$$\xi_0 \stackrel{\cong}{\longrightarrow} \kappa^*(\xi_{A-B}) \longrightarrow \xi_{A-B} \stackrel{\cong}{\longrightarrow} \iota^*(\xi_D) \longrightarrow \xi_D \tag{6}$$

represented in detail in the following commuting diagram:

$$\begin{array}{ccccccccc}
S^2 \times S^1 & \stackrel{\varPhi^{-1}}{\longrightarrow} & P_{\kappa^*(\xi_{A-B})} & \stackrel{pr_2}{\longrightarrow} & \mathbb{C}^* \times S^1 & \stackrel{\varPsi^{-1}}{\longrightarrow} & P_{\iota^*(\xi_D)} & \stackrel{pr_2}{\longrightarrow} & S^3 \\
pr_1 \downarrow & & pr_1 \downarrow & & pr_1 \downarrow & & pr_1 \downarrow & & \downarrow \pi_D \\
S^2 & \stackrel{Id_{S^2}}{\longrightarrow} & S^2 & \stackrel{\kappa}{\longrightarrow} & \mathbb{C}^* & \stackrel{Id_{\mathbb{C}^*}}{\longrightarrow} & \mathbb{C}^* & \stackrel{\iota}{\longrightarrow} & S^2
\end{array}$$

*Diagram 2*

(For simplicity, we omit the upper part of the diagram, involving the right actions $\psi_0$, $\psi^*_{A-B}$, $\psi_{A-B}$, $\psi^*_D$ and $\psi_D$ of $S^1$ on $S^2 \times S^1$, $P_{\kappa^*(\xi_{A-B})}$, $\mathbb{C}^* \times S^1$, $P_{\iota^*(\xi_D)}$ and $S^3$, respectively.)

Since the composition of bundle maps is a bundle map, one obtains the map between the Hopf bundles $\xi_0$ and $\xi_1$:

$$\xi_0 \stackrel{(\alpha, \bar{\alpha})}{\longrightarrow} \xi_1 \tag{7}$$

where

$$\alpha = \iota \circ Id_{\mathbb{C}^*} \circ \kappa \circ Id_{S^2},\ \alpha(b) = z_0 \tag{8}$$

 

and

$$\bar{\alpha} = pr_2 \circ \Psi^{-1} \circ pr_2 \circ \Phi^{-1},\ \bar{\alpha}(b, e^{i\varphi}) = \frac{(z_0, 1)}{||(z_0, 1)||} e^{i\varphi} \tag{9}$$

represented in diagram 3:

$$\begin{array}{ccc}
(S^2 \times S^1) \times S^1 & \overset{\bar{\alpha} \times Id_{S^1}}{\longrightarrow} & S^3 \times S^1 \\
\psi_0 \downarrow & & \downarrow \psi_D \\
S^2 \times S^1 & \overset{\bar{\alpha}}{\longrightarrow} & S^3 \\
pr_1 \downarrow & & \downarrow \pi_D \\
S^2 & \overset{\alpha}{\longrightarrow} & S^2
\end{array}$$

*Diagram 3*

## 4 Conclusion

Eq. (6) or *Diagram 2* clearly exhibit the fact that the bundle corresponding to the $A - B$ effect, $\xi_{A-B}$, is sandwiched between the Hopf bundles corresponding to zero and unit Chern numbers, respectively $\xi_0$ and $\xi_1$. A remarkable fact of this construction is its uniqueness, in the sense that the inclusion $\iota : \mathbb{C}^* \to S^2$ is *canonical* and the map $\kappa : S^2 \to \mathbb{C}^*$ is *unique up to homotopy* (the arbitrariness of the choice of $z_0 \in \mathbb{C}^*$ is irrelevant). This establishes a strong mathematical relation between the $U(1)$-bundles $\xi_0$, $\xi_{A-B}$ and $\xi_D$, and therefore between the $A - B$ effect and the Dirac magnetic monopoles when these are described in the language of fiber bundle theory.

Finally, we want to comment that any smooth or continuous map $\alpha : S^2 \to S^2$, $\eta \to \alpha(\eta)$, induces a bundle morphism $\xi_0 \overset{(\alpha,\bar{\alpha})}{\longrightarrow} \xi_1$ with $\bar{\alpha}(\eta, e^{i\varphi}) = \frac{(\alpha(\eta),1)}{||(\alpha(\eta),1)||} e^{i\varphi}$. This infinite set of morphisms accommodates into an infinite set of homotopy classes indexed by the integers since $\pi_2(S^2) \cong \mathbb{Z}$. The morphism $(\alpha, \bar{\alpha})$ given by (8) and (9) is the one induced by the $A - B$ bundle.

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