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Quantum Integration Using Dirac's Delta Function

Reza Ahangar

Department of Mathematics, Texas A & M University - Kingsville, United States
Email: reza.ahangar@tamuk.edu

Abstract A historical review of Dirac's Delta functions is presented. It is developed as a generalized function in the space Bochner-Lebesgue summable function. Further properties of the absolute, asymptotic continuity, and differentiability of Bochner summable functions is also investigated. We can generalize the equivalent of Bochner-Stieltjes summability and absolute continuity, asymptotic continuity, and differentiability. Dirac Delta sequence of functions will be presented as a generalized function with a convolution operator where we use it as a charaterization of compactness in the space of Bochner summable functions. We will propose a perturbed differential equation such that the perturbation function is a sequence Dirac's Delta function which is a Bochner summable function.

Keywords: Lebesgue Bochner integration, Dirac's Delta function, Compact Operators, Space of Summable Functions, Absolute and Asymptotic Continuity and Differentiability, Nonlinear Operator Differential Equations, impulsive perturbation.

1 Intoduction and History

Paul Dirac intutively presented his original Delta function by

$$\int_{x=-\infty}^{x=\infty} \delta(x)dx = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0 \end{cases} \tag{1.1}$$

although it is not Riemann integrable in the sense of differential and integral calculus and it is not Lebesgue integrable in the sense of measure theory.

Cauchy (1816) and Poisson (1815) derived the Fourier Integral Theorem by using the sifting property of Delta function [1] and [2]. By Fourier Integral theorem:

$$f(x) = \frac{1}{2\pi} \int\limits_{k=-\infty}^{k=\infty} \Big(\int\limits_{\xi=-\infty}^{\xi=\infty} f(\xi) e^{-ik\xi} d\xi \Big) e^{ikx} dk = \int\limits_{k=-\infty}^{k=\infty} f(\xi) \Big(\frac{1}{2\pi} \int\limits_{\xi=-\infty}^{\xi=\infty} e^{-ik(\xi-x)} dk \Big) d\xi \tag{1.2}$$

$$f(x) = \tfrac{1}{2\pi} \int\limits_{k=-\infty}^{k=\infty} \Big(\int\limits_{\xi=-\infty}^{\xi=\infty} f(\xi) e^{-ik\xi} d\xi \Big) e^{ikx} dk = \int\limits_{k=-\infty}^{k=\infty} f(\xi) \Big(\tfrac{1}{2\pi} \int\limits_{\xi=-\infty}^{\xi=\infty} e^{-ik(\xi-x)} dk \Big) d\xi$$

By substituting the relation

$$\tfrac{1}{2\pi} \int\limits_{\xi=-\infty}^{\xi=\infty} e^{-ik(\xi-x)} dk \equiv \delta(\xi - x)$$

in (2) implies that

$$f(x) = \int\limits_{k=-\infty}^{k=\infty} f(\xi)\delta(\xi - x)d\xi, \tag{1.3}$$

a delta function, is the Fourier transformation of a constant unit function [1] *(H.Vic Dannon 2012).*

In a transmission of a signal which lasts a very short time and speed of light in the form of continuous wave will not allow us to measure the time interval between two points.

Dirac's intutive definition (1.1) left open the question, how in the world we can present a delta function undefined at zero $\delta(0)$ and has infinite amplitude? To justify the existence of the integral $\int_{x=-\infty}^{x=\infty} \delta(x)dx$ in definition (1.1) and using the sifting method

$$\lim_{n\to\infty}(\int_{x=-\infty}^{x=-\frac{1}{n}} \delta(x)dx + \int_{x=\frac{1}{n}}^{x=\infty} \delta(x)dx) = 0$$

where it can be accepted that $\lim_{n\to\infty} \int_{x=-\frac{1}{n}}^{x=\frac{1}{n}} \delta(x)dx = 1$.

Laurent Schwartz 1966 *[3] (Shwartz66 , p. 82)* added conditions for auxilary function $\phi(x)$ that is infinitely differentiable and vanishes at $x = -\infty$ and also at $x = \infty$. The Schwartz method also avoids the $\delta(0)$ and leave it an undefined value.

2 Lebesgue-Bochnor-Stieltjes Integral (Bogdon's Approach)

In the following definitions we are presenting just a glimpse of the Bogdanowich integral. This is the Bochner-Steiltjes integral and possibly with modifications could be used for our objectives. For further details see references *[4] Bogdanowicz (1) 65 and [5] Bogdanowicz(2) 65.*

Let R be the space of reals and Y, Z, W be fixed Banach spaces. Denote by U the space of all bilinear continuous operators $u : Y \times Z$ into W. Norms of elements in the space Y, Z, W, and U will be denoted by $|.|$.

Pre-ring: A family of sets of an abstract space X such that

i) $A_1, A_2 \in V$, then $A_1 \cap A_2 \in V$.

ii) If $A_1, A_2 \in V$, then there exist disjoint sets $B_1, B_2, ...B_k \in V$ such that $A_1 \backslash A_2 = B_1 \cup B_2 \cup ...B_k$.

Volume: A function μ from a pre-ring V into a Banach space Z is called volume if it satisfies the following conditions: for all countable disjoint sets $A_t \in V (t \in T)$ such that

$$A = \cup_{t\in T} A_t \in V, \quad \text{we have } \mu(A) = \sum_{t\in T} \mu(A_t), \tag{2.1}$$

where the sum is convergent absolutely and $|\mu|(A) = \sup\{\sum_{t\in T} |\mu(A_t)|\} < \infty$ for any $A \in V$, where the supremum is taken over all possible decompositions of the set into the form (2.1).

Space of Measurable functions: Let us define $M = \{\mu : V \to Z; |\mu(A)| \leq cv(A)$, for all $A \in Z\}$ where $\|\mu\| = \inf\{c : |\mu(A)| \leq cv(A)\}$. It is easy to prove that $(M, \|\mu\|)$ is a Banach space.

Simple Functions: Let us define a set of simple functions

$$S(Y) = \{h : h = y_1\chi_{A_1} + ... + y_k\chi_{A_k}\}, \text{ for all } y_i \in Y, A_i \in V. \tag{2.2}$$

Integral: For a fixed $u \in U, \mu \in M$, define the operator

$$\int u(h, d\mu) = u(y_1, \mu(A_1)) + ... + u(y_k, \mu(A_k)). \tag{2.3}$$

Define also

$$\int hdv = y_1 v(A_1) + ... + y_k v(A_k). \tag{2.4}$$

The operators $\int u(h, d\mu)$ and $\int hdv$ are well defined, that is they do not depend on the choice of function h in (2.2). A sequence of functions $s_n \in S(Y)$ is a **basic** if there exists a sequence $h_n \in S(Y)$ and a cons $\tan t\ M > 0$ such that

$s_n = h_1 + h_2 + ... + h_n$, and $\| h_n \| \leq 4^{-n}M$

for all $n = 1, 2, ...$

The Space of Summable Functions: Let

$L(Y) = \{f : \exists s_n \in S(Y) - basic$ such that $lim\ s_n = f \quad a.e\}$.

Define $\| f \| = \lim \| s_n \|$, $\quad \int u(f, d\mu) = \lim \int u(s_n, d\mu)$, $\quad \int f dv = \lim \int s_n dv$

Further Development: The differentiability and the Bochner-Stieltjes summability need to be studied. Bogdonowicz's approach for Bochner-Stieltjes integral can be used to develop the probabilistic measurable space of stochastic process.

3 Absolute Continuity and Differentiability of Generalized Dynamical Systems

The following theorem which shows the general differentiability in Banach space, has been proved in references *[6] Ahangar 86* and *[7] Ahangar 89.* To use the following results for our objectives one must generalize these differential and integral calculus in the sense of Bochner- Steiltjes integral.

Theorem 3.1: Let f be an **absolute continuous** function in the Lebesgue sense and g a **Bochner summable** function from the interval I into a Banach space Y. The functions f and g satisfy the relation

$$f(x) - f(s) = \int_{[s,x]} g(t)dt \tag{3.1}$$

for all x and s in I, if and only if, the strong derivative $f'(x)$ exists and is equal to $g(x)$ for almost all x in I.

Notice that in this theorem, *absolute continuity* of f and *Bochner summability* of g implies that

$$f\prime(x) = g(x) \qquad \Leftrightarrow \qquad f(x) = f(s) + \int_{[s,x]} g(t)dt \tag{3.2}$$

When the Banach space Y is real then the absolute continuity of f implies the differentiability of function f.

Definition 3.1 (Induced Operator in a Probability Space): Let K be an interval subset in R. Assume that the function f is in Lipschitz space $Lip(K, Y; Z)$ and the induced operator defined by

$$z = F(y) \qquad \Leftrightarrow \qquad z(t) = g(t, y(t)) \qquad \text{for all } t \text{ in } I. \tag{3.3}$$

The following theorems are due to *[6], [7], and [8].*

Theorem 3.2: Let $(Y, | \,|)$ and $(Z, | \,|)$ be Banach spaces and f belongs to Lipschitz space $Lip(K, Y; Z)$. Then the operator F is well defined from $M(K, Y)$ into $M(K, Z)$ and it is Nonanticipative and Lipschitzian.

Theorem 3.3: If y is a function in the initial domain $D(\phi, Y)$ and $z \in M(I, Y)$, then the following two relations are equivalent $y\prime(t) = z(t)$ for almost all t in I

$$\Leftrightarrow \qquad \begin{cases} y(t) = \phi(0) + \int_0^t z(s)ds & \text{for almost all } t \text{ in } I \\ \phi(t) & \text{for a.a } t < 0. \end{cases} \tag{3.4}$$

Remarks:

(i) If the integrand z in Theorem 3.3 is an induced operator generated by a function g, then the integral equation will be in the following form

$$y\prime(t) = g(t, y(t)) \qquad \Leftrightarrow \qquad y(t) = \phi(0) + \int_0^t g(s, y(s))ds \tag{3.5}$$

provided that $g \in Lip(I, Y; Z)$.

(ii) Can we modify the Theorem (3.3) when y is absolutely continuous and z is Bochner-Stieltjes summable? In other words

$$dy(t) = g(t, y(t))dt + z(t)dw(t) \tag{3.6-a}$$

$$\Leftrightarrow y(t) = \phi(0) + \int_0^t g(s, y(s))ds + \int_0^t z(s)dw(s) \tag{3.6-b}$$

for almost all t in I.

Further Studies: This is absolute continuity and differentiability of Bochner summable functions. We can generalize the equivalent Bochner-Stieltjes version of absolute continuity and differentiability.

4 Definition of Uniformly Continuous

A subset Q of a metric space (X, d) is said to be a **compact** set if and only if every open cover of Q has a finite subcover. Let Q be a compact subset of a metric space (X, d) and let (Y, d) be a complete metric space.

Then every continuous map

$$K : Q \subset (X, d) \to (Y, d) \tag{4.1}$$

K from Q to Y is **uniformly continuous**. That is for every $\in> 0$ there exists $d_0 > 0$ such that whenever $d(x, y) < d_0$ where x and y are in Q then

$$d(K(x), K(y)) <\in . \tag{4.2}$$

A family $\{K_t : t \in T\}$ of operators is said **equicontinuous** if and only if, for every $\in> 0$ there exists $d_0 > 0$ such that whenever $d(x, y) < d_0$, where x and y are in Q, then

$$d(K_t(x), K_t(y)) <\in \tag{4.3}$$

for all $t \in T$.

Let R^+ be the extended closed interval $[0, \infty]$.

A function h from R^+ into R^+ will be called a **modulus** if and only if, (a) $h(0) = 0$ and (b) the function h is continuous at the point zero and nondecreasing.

Modulus of Continuity: A function h will be called a **modulus of continuity** for a family $\{Kt : t \in T\}$ of operators, if and only if, the function h is a modulus and

$$d(K_t(x), K_t(y)) \leq h(d(x, y)) \tag{4.4}$$

for all $t \in T$ and all $x, y \in Q$.

In the following theorem one can show that: *Equicontinuity* is equivalent to the *Existence of Modulus of Continuity.*

Proposition 4.1: Let Q be a nonempty subset in a metric space (X, d). A family $F = \{K_t : t \in T\}$ of operators from Q into a metric space (Y, d) is equicontinuous, if and only if, there exists a modulus of continuity w for the family F of operators.

Proposition 4.2: Let (X, d) be a metric space and let (Y, d) be a complete metric space. Let Q be a dense subset of X. Let g_n be a sequence of functions from X into Y. Let the family $\{g_n : n \in N\}$ be equicontinuous. If the limit $lim_n g_n(x)$ exists for every $x \in Q$, then the limit

$lim_n g_n(x)$ exists for every $x \in X$.

Convolution Operation: Let (X, V, v) be a volume space. Let (Y,$|\cdot|$) be a Banach space and $L(v, Y)$ denote as before the space of Bochner summable functions f from X into Y generated by the volume v.

Let $BM(v, Y)$ denote the space of all Bochner measurable functions. This space can be characterized *[9] (Bogd66(l), Theorem 1, p. 253)* as the space of functions where they are the limit of sequences of simple functions, v-almost everywhere. Let $M^+(v)$ denote the space of extended measurable functions f from R into the closed interval $[0, \infty]$ This space of functions can be characterized *[9] (Bogd66(l), Theorem 2, p. 255)* as the space of all functions f such that there exists an increasing sequence of nonnegative functions $f_n \in L(v, R)$, for which the sequence of values $f_n(x)$ tends to $f(x)$, v-almost everywhere.

Let (R, V, v) denote the Riemann volume space. We shall assume that there are given two copies (R, V_1, v_1) and (R, V_2, v_2) of this space. Let (R^2, W, w) be the product volume space of these spaces, that is

$$W = \{I_1 \times I_2 : I_1 \in V_1 \text{ and } I_2 \in V_2\}$$

and $w(I_1 \times I_2) = v_1(I_1)v_2(I_2)$ for all $I_1 \times I_2 \in W$. It is easy to prove that W is a prering and w is a volume on it *[10] (see Bogd67, p. 236, Theorem 3, part 2).*

Let $f \in L(v, Y)$ and $g \in L(v, R)$. Define the **convolution operation** by means of the formula

$$(f * g)(x) = \int_R f(s)g(x - s)v(ds) \tag{4.5}$$

for all x for which the integral exists.

We shall prove the following theorem.

Theorem 4.1: If $f \in L(v, Y)$ and $g \in L(v, R)$ then the convolution

$$(f * g)(x) = \int f(s)g(x - s)v(ds)$$

is well defined for almost all x and

$$f * g \in L(v, Y).$$

Moreover

$$\|f * g\| \leq \|f\| \, \|g\| \, .$$

Definition 4.1: Dirac's Delta Sequences: By a sequence of functions convergent to *Dirac's delta function*, we shall mean a sequence of real-valued functions a_k satisfying the following properties:

(1) Each function a_k is infinitely differentiable on the space R of reals;

(2) $a_k(x) \geq 0$ for all x in R;

(3) $a_k(x) = 0$ if x is not in $[-1/k, 1/k]$;

(4) the integral $\int_R a_k(x)dx = 1$.

To construct such a sequence we can start with a nonnegative function $a(x)$, infinitely differentiable, having support in the interval $[-1, 1]$ and such that the integral

$$\int_R a(x)dx = 1.$$

Put $a_k(t) = a(x/k)$ for all $x \in R$ and all $k > 0$ *[2] (see Dirac35).*

From now on the **volume** in the integrals will be understood as the Riemann volume and we will write dy instead of $v(dy)$. The symbol $CB(R, Y)$ will denote the Banach space of all continuous bounded functions from R into Y. The norm in this space is the supremum norm.

Convolution Operator on a Sequence of Functions: Define operators A_k according to the definition of convolution (4.5) on the space $L(v, Y)$ by the formula

$$A_k f = a_k * f \tag{4.6}$$

for all $f \in L(v, Y)$.

Proposition 4.3: The operator A_k is linear and continuous from $L(v, Y)$ into the space $CB(R, Y)$ continuous bounded functions with the supremum norm.

Proof: We want to prove that the operator A_k is linear. Assume a and b are two constant numbers and f and g are in the space $L(v, Y)$. By definition of the operator A_k we have that

$$\begin{gathered} A_k(af + bg)(x) \\ = \int a_k(u)(af + bg)(x - u)du \\ = a \int a_k(u)f(x - u)du + b \int a_k(u)g(x - u)du \\ = aA_k(f)(x) + bA_k(g)(x) \end{gathered}$$

This proves that the operator A_k is linear.

Let $g = A_k f$ for a fixed but arbitrary $f \in L(v, Y)$. We need to prove that the function g is continuous from R into Y.

Take a sequence $x_n \in R$ convergent to a point x_0. Consider the values $g(x_n)$. By definition of the convolution

$$g(x_n) = (A_k f)(x_n) = \int f(s)a_k(x_n - s)ds.$$

Let us denote $h_n(s) = f(s)a_k(x_n - s)$. Since x_n tends to x_0, the sequence $x_n - s$ tends to $x_0 - s$. Therefore

$$a_k(x_n - s) \text{ approaches } a_k(x_0 - s).$$

Since the function a_k is bounded, we get the estimate

$$|h_n(s)| \leq c. |f(s)|$$

for all n and $s \in R$ and some $c > 0$. Thus from the dominated convergence theorem, *[11] and [12](see Bogd65, Theorem 5, p. 497)* we get that one can pass to the limit under the integration sign. Thus the sequence $g(x_n)$ tends to

$$\int f(s)a_k(x_0 - s)ds = g(x_0).$$

This proves that the function g is continuous.

To prove that the operator A_k is bounded, notice the estimate

$$\|A_k f\|_0 \leq \int |f(s)| .c.ds = c \int |f(s)| ds = c \|f\|.$$

Hence the operator A_k is continuous from the space $L(v, Y)$ into $CB(R, Y)$.

In the following theorem let T denote the identity operator on the space $L(v, Y)$, that is the operator defined by the formula

$$T(f) = f$$

for all $f \in L(v, Y)$.

Theorem 4.2: The sequence of operators A_k converges pointwise to the operator T, in the topology of the space $L(v, Y)$, that is for every $f \in L(v, Y)$ we have

$$lim_k A_k(f) = T(f) = f.$$

Proof: The operators A_k are linear from $L(v, Y)$ into $L(v, Y)$ as follows from the properties of the convolution operation. Notice also that

$$\|A_k f\| = \|f * a_k\| \leq \|a_k\| . \|f\|$$

for all $f \in L(v, Y)$. Hence, from the definition of the functions a_k, we have

$$\|a_k\| = \int a_k(x)dx = 1,$$

and the operators A_k are equicontinuous on $L(v, Y)$. Thus to prove that they converge at every point of the space $L(v, Y)$ it is enough to show that they converge at every point of some set $S \subset L(v, Y)$ which is dense in the space $L(v, Y)$. Let S be the set $S(V, Y)$ of simple functions. The set of simple functions, as follows from the definition of the space $L(v, Y)$ and *[12] (Lemma 4, p. 495, Bogd65 (2))*, is dense in the space $L(v, Y)$ of summable functions. Indeed, take any function $f \in L(v, Y)$. There exists a basic sequence s_n convergent v-almost everywhere to the function f. By the lemma we have $\|s_n - f\| <\in$ if $n > k$. This implies that in every ball $B(f, \in)$ of radius $\in$ around the point f, one can find a simple function $s_n \in S(V, Y)$. The set of simple functions is dense in the space $L(v, Y)$.

Since the operators A_k are linear and every simple function can be written as sum of a finite number of the functions of the form

$$s = C_I Y \tag{4.7}$$

where $I \in V$ and $y \in Y$, it is sufficient to prove that the sequence $A_k(s)$ converges in the norm of the space $L(v, Y)$ to $s(x)$ for every function s of the form (4.7). To this end assume that the function$s(x)$ is fixed. Let $I = (a, b]$and $B = (a - 2, b + 2]$. Let $g_k = A_k s$. From the definition of the operator A_k we have

$$\begin{gathered} |g_k(x)| = |(s * a_k)(x)| \\ \leq \left| \int s(u)a_k(x - u)du \right| \\ \leq \int |a_k(x - u)| |y| C_I(u)du \\ \leq |y| \end{gathered}$$

for all $x \in R$. Notice also that the function g_k vanishes outside of the interval B. Thus we get the estimate

$$|g_k(x)| \leq |y|\, C_B(x)$$

for all $x \in R$. It is easy to see that the sequence g_k converges at each point $x \neq b$ and $x \neq a$ to the value $C_I(x)y$. This implies that the sequence g_k converges to the function $C_I y$ v-almost everywhere. Thus from the dominated convergence theorem, that is *[12]Theorem 5, p. 497, Bogd65 (2),* we can deduce that the sequence g_k converges to the function s in the topology of the space $L(v, Y)$. Q.E.D.

5 Characterization of Compactness in the Space L(v,Y) of Bochner Summable Functions

Let a_k be a sequence of functions as defined in (4.1). Let A_k be the operator from $L(v, Y)$ into the space $CB(R, Y)$ defined by the formula

$$A_k(f) = f * a_k \qquad \text{for all } f \in L(v, Y) \tag{5.1}$$

and for every $k > 0$ where

$$(f * a_k)(x) = \int_R f(s) a_k(x - s) v(ds) \tag{5.2}$$

for all x in which the integral exists.

In the following we shall establish a characterization of compactness in the space of summable functions. Let I be a compact interval.

Notice that the space L of Lebesgue-Bochner summable functions over the interval I can be identified with the set

$$L = \{f \in L(v, Y) : f(x) = 0 \text{ if } x \notin I\}.$$

The topology in the space L is defined as usual as the topology generated-by the seminorm

$$\|f\| = \int |f|\, dv.$$

Notice that the set L is closed in the space $L(v, Y)$ as follows from *[12] (Bogd65 (2), Theorem 2, p.496).*

Proposition 5.1: A subset E of L is relatively compact, if and only if, the following two conditions are satisfied

(1) the image $A_k(E)$ is relatively compact in the space $C(I, Y)$ for every k,

(2) $\lim_k A_k f = f$ uniformly for $f \in E$ in the seminorm topology of the space L.

Proof: Assume that the set E is a relatively compact subset of the space L. Since the set E is relatively compact, its closure $\overline{E}$ is compact. Since the operator A_k is continuous, the image $A_k(\overline{E})$ is compact. Therefore $A_k(E)$ is relatively compact. This proves the necessity of condition (1).

Since the set E is relatively compact, for every $\in > 0$ there exists a finite e-net in E. In other words for every $\in > 0$ there exists $f_1, f_2, ..., f_n \in E$ such that, the union $\cup_i S(f_i, \in)$ of balls contains E where the union is over the set $\{i : i = 1, ..., n\}$

Let $k_0 > 0$ be such that

$$\|f_i * a_k - f_i\| < \in$$

if $k > k_0$ and $i = 1, 2, ..., n$. Such k_0 exists because $\|f_i * a_k - f_i\|$ converges to zero by Theorem 4.2 Take any $f \in E$. There exist a ball $S(f_j, \in)$ containing f. By definition of a ball we have $\|f - f_j\| < \in$. Now by the triangle inequality

$$\|f * a_k - f\| \leq \|f * a_k - f_j * a_k\| + \|f_j * a_k - f_j\| + \|f_j - f\|$$

Notice that

$$\|f * a_k - f_j * a_k\| = \|(f - f_j) * a_k\| = \|a_k\|\, \|f - f_j\| = \|f - f_j\|$$

Therefore the above inequality yields

$$\begin{aligned}\|f * a_k - f\| &\leq 2\,\|f - f_j\| + \|f_j * a_k - f_j\| \\ &\leq 2 \in + \|f_j * a_k - f_j\| \\ &\leq 3 \in\end{aligned}$$

for all $k > k_0$ and $f \in E$.

This implies that operators $A_k f = f * a_k$ tend to f uniformly on E.

Therefore, if the set E is **relatively compact** in the space of $L(v, Y)$, then the conditions (1) and (2) hold.

Conversely let us assume that the conditions (1) and (2) are satisfied. Suppose E is a subset of L. We want to prove E is relatively compact in L. By (2), the operator $A_k f$ approaches f uniformly on the set E when k approaches ∞. Hence, for any $\in > 0$ there exists $k_0 > 0$, such that

$$\|A_k f - f\| \leq \in \text{ for all } f \in E \text{ and } k \leq k_0.$$

By (1) the set $A_k(E)$ is a relatively compact subset of the space $C(I, Y) = C$. Therefore for every $\in > 0$ there exists a finite $\in$–net, i.e. there exist $g_1, g_2, ..., g_n \in A_k(E)$ such that

$$\cup_i S(g_i, \in) \supset A_k(E)$$

where the union is taken over $i = 1, ..., n$. Thus, for every $f \in E$ there is g_i such that

$$\|A_k f - g_i\|_0 < \in \tag{5.3}$$

By definition of the image set for every $g_i \in A_k(E)$, there is $f_i \in E$ such that

$$A_k(f_i) = g_i.$$

From the relation (1), we get the following inequality

$$\|A_k f - A_k f_i\|_0 \leq \in .$$

From the triangle inequality we can write

$$\|f - f_i\| \leq \|f - A_k f\| + \|A_k f - A_k f_i\| + \|A_k f - f_i\|$$

Notice that for any continuous function g we have the estimate

$$\|g\| = \textstyle\int_I |g(x)|\, dx \leq \|g\|_0 .v(I).$$

Notice the estimate

$$\|A_k f - A_k f_i\| = \|A_k(f - f_i)\| = \|(f - f_i) * a_k\| = \|a_k\| . \|f - f_i\| = \|f - f_i\|$$

Hence,

$$\|f - f_i\| < 2 \in + \|A_k f - A_k f_i\| \leq 2 \in + \|A_k f - A_k f_i\|_0 .v(I).$$

and the previous inequality will yield

$$\|f - f_i\| \leq 2 \in + \in .v(I) = \in (2 + v(I)).$$

We have proved that for every $\in > 0$ there exists a finite $d(\in)$-net for the set E, where

$$d(\in) = \in (2 + v(I)).$$

This is equivalent to the property that for every $\in > 0$ there exists a finite $\in$-net for the set E. Since there exists a finite $\in$-net for the set E, the set E is relatively compact according to Hausdorff's characterization of compactness in the metric space. Q.E.D.

6 Applications of Dirac Delta Functions to Impulsive Perturbation

Very often random fluctuations or different noises in the physical systems requires us to use differential equations with random perturbation. The introductory model of (3.6-a) and (3.6-b) are well known models. If the perturbations follow the brownian motion or Wienner process, then the well known stochastic differential equations can be studied by a variety of stochastic integral calculus *[13] (ahangar 2013) and [14] (ahangar 2010)* .

Example 6.1 (Langevin Equation): Consider the motion of a particle of mass m immersed in a fluid. Let $x(t)$ denote the position of the particle at time t, and let $v(t)$ denote its velocity. The medium surrounding the particle offers resistance to the motion of the particle which is in the form of a frictional force equal to $\beta v(t)$, where β is the (mean) dynamic frictional coefficient *[15] (Langevin 1908).* The fluctuations in the number of collisions of molecules of the fluid with the particle appear as a random force $F(t)$. The equation of the motion of the particle is of the form

$$m\frac{dv}{dt} = -\beta v(t) + F(t). \tag{6.1}$$

where $\alpha = \beta/m$. For the rigorous mathematical treatment of the Langevin equation *[16] (Chandrasekhar 1943) and [17] (Doob 1942)* .

In Newtonian Mechanics the reaction of the deterministic forces are instantaneous. Let us pretend that the random force is acting with a constant delay r. The probabilistic form of this differential equation when, $F(t-r,\omega)$ is a random force will be

$$dv(t,\omega) = -\alpha v(t,\omega)dt + dF(t-r,\omega). \tag{6.2}$$

Example 6.2 (Stochastic Verhulst Model): Assume that the random effect of the Verhulst logistic population model can be obtained by adding a random input term signifying the combined effect of all the exogenous factors which affect the population, such as food, climate, and predators. Assume that the intensity of the random input at any instant of time is proportional to the value of the population at that time. Thus the stochastic version will be

$$y\prime(t) = ry(t)(1 - \frac{y(t)}{K}) + y.w(t). \tag{6.3}$$

Dividing each side by y and using the derivative of $\ln(y)$ we can approximate the solution. If δ is a small interval of time then

$$\ln y(t+\delta) - \ln y(t) = \delta[r - (r/K)y(t)] + w(t+\delta).$$

For zero mean and standard deviation $\delta = 1$, we obtain the following stochastic difference equation,

$$\ln y(t) - \ln y(t-1) = r - (r/K)y(t-1)] + w(t). \tag{6.3}$$

For more example of this type see *[18] (Kashyap 1976).*

Example 6.3 - Stochastic Differential Equation and Random Impulse Perturbation:

Ito's integral and ito's calculus are historical attempts to solve problems presented in the above two examples. In a classical definition of solution to the differential equations

$$y\prime(t) = f(t, y(t)), \qquad y(t_0) = y_0 \tag{6.4}$$

the function $y(t)$ is said to be a solution of the differential equation when it satisfies the system. When $y(t)$ is the solution to the system (6.4) it will satisfy the equivalent integral system

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds. \tag{6.5}$$

The idea of strong and weak solution is an important factor in introducing the definition of solution to the stochastic differential equations. Ito introduced his integration using weak solution.

That is we call a random function $y(t, \omega)$ a solution to the system of stochastic differential equation

$$dy(t, \omega) = f(t, y(t, \omega))dt + g(t, y(t, \omega))dw(t, \omega) \tag{6.6}$$

if it satisfies the integral

$$y(t, \omega) = y_0(\omega) + \int_{t_0}^{t} f(\tau, y(\tau, \omega))d\tau + \int_{t_0}^{t} g(\tau, y(\tau, \omega))dw(\tau, \omega). \tag{6.7}$$

When $w(t, \omega)$ is smooth, then two systems (6.6) and (6.7) are equivalent. Ito introduced a model for nonsmooth solution to the stochastic system of equation (6.6). An equation of the form (6.7), called **Ito's stochastic integral equation**, may not be equivalent to the random differential equation (6.3) (*see [19] Arnold 74 and [20] Baharucha 72).*

In certain conditions when the above relations (6.6) and (6.7) are equivalent, then the system (6.6) is called **Ito's stochastic differential equation**.

Let us denote Ito's integral of the function g by *I(g)* where

$$I(g)(\omega) = \int_0^1 g(s, \omega)dW_s(\omega) = \sum_{j=1}^{n} g_j[W_{t_{j+1}}(\omega) - W_{t_j}(\omega)] \tag{6.8}$$

for a nonrandom step function $g(t, \omega) = g_i$ on the subinterval $t_j \leq t \leq t_{j+1}$ for $j = 1, 2, ..., n$. The integrand is nonanticipative and W is a Wiener process. However, Ito's integral also has a peculiar property, amongst others, that

$$\int_0^t W_s(\omega)dW_s(\omega) = \frac{1}{2}W_t^2(\omega) - t/2 \tag{6.9}$$

$w.p.1$. In the evaluation of the integral of the formula (6.5) we evaluate the function on the left hand point of each subinterval. Evaluation of the function at the middle of the discretization intervals leads to the definition of the **Stratonovich Integral** and it is denoted by "$\circ$". Unlike the Ito integral, the **Stratonovich integral** has properties similar to standard calculus, for example,

$$\int_0^t W_s(\omega) \circ dW_s(\omega) = \frac{1}{2}W_t^2(\omega) \tag{6.10}$$

Unfortunately, the Stratonovich integral does not have the martingale property which is crucial in stochastic analysis. We are searching for a unified analytical approach for solutions to stochastic differential equations. The McShane approach is a very diverse method toward unification of integrals does not provide the desired technique for stochastic differential equations (*see [21] McShane 83 and [22] McShane 74*).

Now there are many questions which arise on this matter. i) Could we choose $w(t)$ a sequence of infinitely differentiable functions instead of nowhere differentiable functions? ii) Could we use the idea of delta function to generalize the concept of the pertubation and re-establish a new workable integration system?

Conclusion: A general form of the Dirac's Delta functions is defined as a real valued nonnegative sequence of functions a_k, infinitely differentiable on the space R, and are zero outside of the interval $\left[\frac{-1}{k}, \frac{1}{k}\right]$ where the integral $\int_R a_k(x)dx = 1$.

For any function f belonging to the space of Bochner summable functions $L(v, Y)$ and $a_k \in L(v, R)$ define a convolution operator

$$(f * a_k)(x) = \textstyle\int_R f(x)a_k(x - s)v(ds).$$

In Theorem (4.1) it was proved that the convolution $A_k(f) = f * a_k \in L(v, Y)$ is a well defined and linear continuous bounded operator. This is a property of the Bochner summable functions that they are convolved and one of them is a Dirac Delta sequence.

In the Theorem 5.1 we expanded the properties of the convolution operation $A_k(f) = f * a_k$ to a relatively compact subset $E \subset L = L(v, Y)$. The image $A_k(E)$ is relatively compact in $C(I, Y)$ and with a limit of $f \in E$.

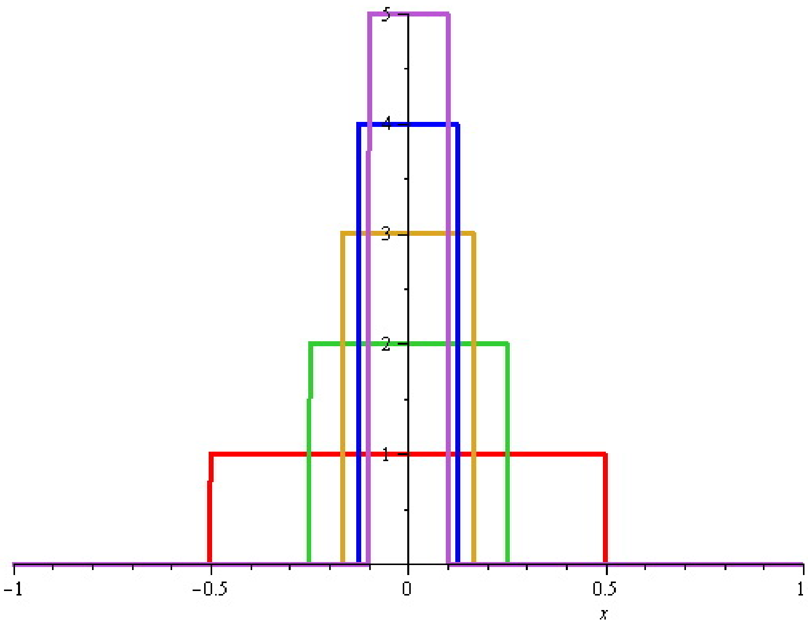

Figure 1. Sequence of rectangular impulse functions approaching Dirac's Delta function.

7 Future Plan: Integration System for Modeling Impulse Random Perturbation

Establish a new integration system that

i) is consistent with the standard differential and integral calculus,

ii) can be easily generalized to vectorial functional space without abandoning the Riemann (or Riemann-Steiltjes) integrals.

iii) the general Bochner-Stieltjes integral in Banach space will be applicable to the stochastic calculus.

iv) will provide a concise method of solving stochastic differential equations.

v) provides a consistent algorithmic and numerical approach toward the solution to stochastic differential equations.

In the following examples, we used MAPLE computer algebra system (CAS) to demonstrate a sequence of functions approaching Dirac's Delta function.

Impulse Example (1): Rectangular impulse sequence of Functions approaching to Dirac's Delta function:

```
> restart;
> with(plots);
> a := proc (x) options operator, arrow; piecewise(abs(x) <= (1/2)/n, n, (1/2)/n < x, 0) end proc;
> plot({seq(a(x), n = 1 .. 5)}, x = -1 .. 1, thickness = 3);
```

Example (2): Verify that the following sequence functions satisfy the definition of Dirac's Delta sequence. $f_n(x) = \dfrac{n}{2[\cosh(nx)]^2}$.

The following Maple program generates a sequence of impulse functions satisfying the definition of Dirac's Delta functions.

```
> restart;
> with(plots);
> B := proc (x) options operator, arrow; (1/2)*n/cosh(n*x)^2 end proc;
> plot({seq(B(x), n = 0 .. 5)}, x = -4 .. 4, thickness = 2); n := 'n';
```

Impulse Example (3): It can be verified that the following sequence of functions $\{f_n\}$ approaches to Dirac's Delta functions when $n \to \infty$:

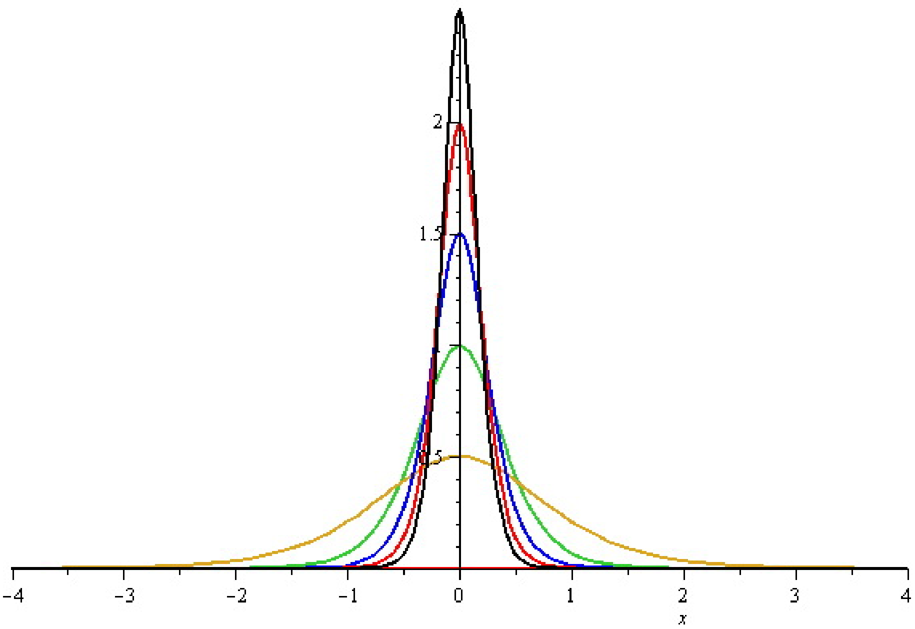

Figure 2. Sequence of impulse functions approaching to delta function.

$$f_n(x) = \begin{cases} 0 & if \quad x \le a \quad or \quad x \le b + \frac{\in}{n} \\ 1 & if \quad a + \frac{\in}{n} \le x \le b, \\ \frac{n}{\in}(x-a) & if \quad a < x < a + \frac{\in}{n}, \\ -\frac{n}{\in}(x-b)+1 & if \quad b < x < b + \frac{\in}{n}, \end{cases}$$

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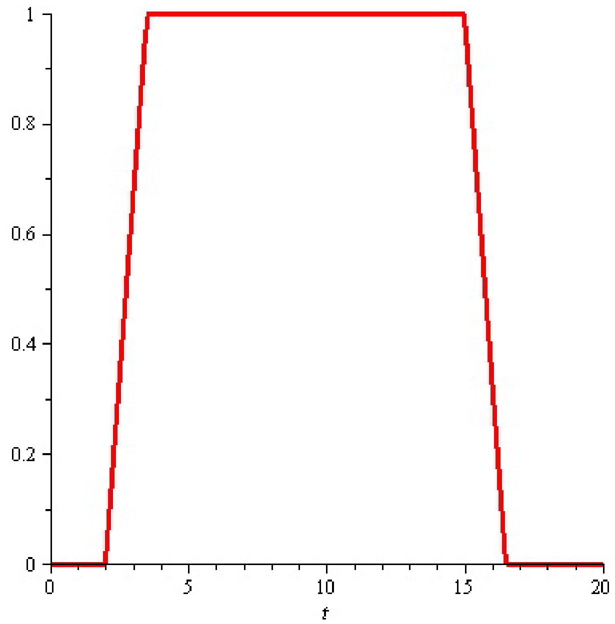

Figure 3. This is a sequence of functions $f_n(x)$ approaching Dirac's Delta sequence.

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