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Lebesgue Function for Higher Order Hermite-Fejér Interpolation Polynomials with Exponential-Type Weights

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Abstract Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in \mathbf{C}^1(\mathbb{R}) : \mathbb{R} \to [0, \infty)$ be an even function which is an exponent. We consider the weight $w(x) = e^{-Q(x)}$, $x \in \mathbb{R}$ and then we can construct the orthonormal polynomials $p_n(w^2; x)$ of degree n for $w^2(x)$. In this paper, we study the (l, ν) order Hermite-Fejér interpolation polynomial $L_n(l, \nu, f; x)$ based on the zeros $\{x_{k,n}\}_{k=1}^n$ of $p_n(w^2; x)$, and we estimate the Lebesgue function of $L_n(l, \nu, f; x)$.

Keywords: higher order Hermite-Fejér interpolation polynomial, Lebesgue function

1 Introduction

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in \mathbf{C}^1(\mathbb{R}) : \mathbb{R} \to [0, \infty)$ be an even function. We consider the weight $w(x)$;

$$w(x) := \exp(-Q(x)), x \in \mathbf{R}.$$

Then we suppose that $\int_0^\infty x^n w^2(x)dx < \infty$ for all $n = 0, 1, 2, \ldots$. Now we can construct the orthonormal polynomials $p_n(x) = p_n(w^2; x)$ of degree n for $w^2(x)$, that is,

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)w^2(x)dx = \delta_{mn} \quad \text{(Kronecker delta)}.$$

We denote the zeros of $p_n(x)$ by

$$-\infty < x_{n,n} < x_{n-1,n} < \ldots < x_{2,n} < x_{1,n} < \infty.$$

For $f \in C(\mathbb{R})$ we define the higher order Hermite-Fejér interpolation polynomial $L_n(\nu, f; x)$ based on the zeros $\{x_{k,n}\}_{k=1}^n$ as follows:

$$L_n^{(i)}(\nu, f; x_{k,n}) = \delta_{0,i} f(x_{k,n}),\ k = 1, 2, \ldots, n,\ i = 0, 1, \ldots, \nu - 1.$$

$L_n(1, f; x)$ is the Lagrange interpolation polynomial, $L_n(2, f; x)$ is the ordinary Hermite-Fejér interpolation polynomial, and $L_n(4, f; x)$ is the Krylov-Stayermann polynomial. The fundamental polynomials $h_{k,n}(\nu; x) \in \mathcal{P}_{\nu n-1}$, where we denote the class of polynomials with degree n by $\mathcal{P}_n$, for the higher order Hermite-Fejér interpolation polynomial $L_n(\nu, f; x)$ are defined as follows:

$$h_{k,n}(\nu; x) = l_{k,n}^{\nu}(x) \sum_{i=0}^{\nu-1} e_i(\nu, k, n)(x - x_{k,n})^i,$$

$$l_{k,n}(x) = \frac{p_n w^2; x)}{(x - x_{k,n})p_n'(w^2; x_{k,n})},$$

$$h_{k,n}(\nu; x_{p,n}) = \delta_{kp}, \quad h_{k,n}^{(i)}(\nu; x_{p,n}) = 0, \quad k, p = 1, 2, \ldots, n,\ i = 1, 2, \ldots, \nu - 1.$$

Using them, we can write as follows:

$$L_n(\nu, f; x) = \sum_{k=1}^{n} f(x_{k,n})h_{k,n}(\nu; x).$$

Furthermore, we extend the operator $L_n(\nu, f; x)$. Let l be a non-negative integer, and let $\nu - 1 \geqslant l$. For $f \in \mathbf{C}^l(\mathbb{R})$ we define the (l, ν)-order Hermite-Fejér interpolation polynomials $L_n(l, \nu, f; x) \in \mathcal{P}_{\nu n-1}$ as follows: For each $k = 1, 2, \ldots, n$,

$$\begin{aligned} &L_n(l, \nu, f; x_{k,n}) = f(x_{k,n}), \\ &L_n^{(j)}(l, \nu, f; x_{k,n}) = f^{(j)}(x_{k,n}),\ j = 1, 2, \ldots, l, \\ &L_n^{(j)}(l, \nu, f; x_{k,n}) = 0,\quad j = l+1, l+2, \ldots, \nu - 1. \end{aligned}$$

Especially $L_n(0, \nu, f; x)$ is equal to $L_n(\nu, f; x)$, and for each $P \in \mathcal{P}_{\nu n-1}$ we see $L_n(\nu - 1, \nu, P; x) = P(x)$. The fundamental polynomials $h_{s,k,n}(\nu; x) \in \mathcal{P}_{\nu n-1},\ k = 1, 2, \ldots, n$, of $L_n(l, \nu, f; x)$ are defined by

$$\begin{aligned} &h_{s,k,n}(\nu; x) = l_{k,n}^{\nu}(x) \sum_{i=s}^{\nu-1} e_{si}(\nu, k, n)(x - x_{k,n})^i, \\ &h_{s,k,n}^{(j)}(\nu; x_{p,n}) = \delta_{sj}\delta_{kp},\ j, s = 0, 1, \ldots, \nu - 1,\ p = 1, 2, \ldots, n. \end{aligned}$$

Then we have

$$L_n(l, \nu, f; x) = \sum_{k=1}^{n} \sum_{s=0}^{l} f^{(s)}(x_{k,n}) h_{s,k,n}(\nu; x).$$

In this paper we estimate the Lebesgue function of $L_n(l, \nu, f; x)$. Then we give an application with respect to the uniform convergence of $L_n(l, \nu, f; x)$.

For any nonzero real valued functions $f(x)$ and $g(x)$, if there exist constants $C_1,\ C_2 > 0$ independent of x such that $C_1 g(x) \leqslant f(x) \leqslant C_2 g(x)$ for all x in the range, then we write $f(x) \sim g(x)$. Similarly, for any two sequences of positive numbers $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{=1}^{\infty}$ we define $c_n \sim d_n$.

Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t or polynomials $P_n(x)$. The same symbol does not necessarily denote the same constant in different occurrences.

We say that $f : \mathbb{R} \to [0, \infty)$ is quasi-increasing if there exists $C > 0$ such that $f(x) \leqslant Cf(y)$ for $0 < x < y$.

First we need the following definition from [6].

Definition 1.1. The weight $w(x) = \exp(-Q(x))$ satisfies the following. Let $Q : \mathbb{R} \to [0, \infty)$ be a continuous and an even function, and satisfy the following properties:

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R}\backslash\{0\}$.

(c)

$$\lim_{x\to\infty} Q(x) = \infty.$$

(d) The function

$$T(x) := T_w(x) := \frac{xQ'(x)}{Q(x)},\quad x \neq 0$$

is quasi-increasing in $(0, \infty)$, with

$$T(x) \geqslant \Lambda > 1,\quad x \in \mathbb{R}\backslash\{0\}.$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leqslant C_1 \frac{|Q'(x)|}{Q(x)},\quad a.e.\ x \in \mathbf{R}\backslash\{0\}.$$

Then we write $w = \exp(-Q) \in \mathcal{F}(C^2)$. If there also exists a compact subinterval $J(\ni 0)$ of $\mathbb{R}$, and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geqslant C_2 \frac{|Q'(x)|}{Q(x)},\quad a.e.\ x \in \mathbb{R}\backslash J,$$

then we write $w = \exp(-Q) \in \mathcal{F}(C^2+)$.
Example 1.2. (1) If an exponential $Q(x)$ satisfies

$$1 < \Lambda_1 \leqslant \frac{(xQ'(x))'}{Q'(x)} \leqslant \Lambda_2,$$

where $\Lambda_i,\ i = 1, 2$ are constants, then we call $w = \exp(-Q(x))$ the Freud-type weight. The class $\mathcal{F}(C^2+)$ contains the Freud-type weights.
(2) (cf. [2]) For $\alpha > 1,\ r \geqslant 1$ we define

$$Q(x) = Q_{r,\alpha}(x) = \exp_r(|x|^\alpha) - \exp_r(0),$$

where $\exp_l(x) = \exp(\exp(\exp \ldots \exp x) \ldots)$ (r times). Moreover, we define

$$Q_{r,\alpha,m}(x) = |x|^m\{\exp_r(|x|^\alpha) - \alpha^* \exp_r(0)\}, \alpha + m > 1, m \geqslant 0, \alpha \geqslant 0,$$

where $\alpha^* = 0$ if $\alpha = 0$, and otherwise $\alpha^* = 1$. We note that $Q_{r,0,m}$ gives a Freud-type weight.
(3) We define

$$Q_\alpha(x) = (1 + |x|)^{|x|^\alpha} - 1, \alpha > 1.$$

If w is a Freud-type weight, then we see that $T(x)$ is bounded. If $T(x)$ is unbounded, then we call w the Erdös-type weight.
Notation 1.3. We use the following notations.
(1) Mhaskar-Rakhmanov-Saff numbers a_x;

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{(1 - u^2)^{1/2}} du, x > 0.$$

(2)

$$\varphi_u(x) = \begin{cases} \frac{a_u}{u} \frac{1 - \frac{|x|}{a_{2u}}}{\sqrt{1 - \frac{|x|}{a_u} + \delta_u}}, & |x| \leqslant a_u; \\ \varphi_u(a_u), & a_u < |x|, \end{cases}$$

$$\delta_u = \{uT(a_u)\}^{-2/3}.u > 0.$$

We define

$$\Phi_n(x) := \max\{1 - \frac{|x|}{a_n}, \delta_n\} \tag{1.1}$$

and

$$\Phi(x) := \frac{1}{(1 + Q(x))^{2/3} T(x)}.$$

Here we note that for $0 < d \leqslant |x|$,

$$\Phi(x) \sim \frac{Q(x)^{\frac{1}{3}}}{xQ'(x)}.$$

We have the following.
Lemma 1.4. [5, Lemma 3.4] For $x \in \mathbb{R}$ we have

$$\Phi(x) \leqslant C\Phi_n(x), n \geqslant 1.$$

Theorem 1.5. Let $w \in \mathcal{F}(C^2+)$, and let ν be a positive integer. Then we have the following. For $x \in \mathbb{R}$, we have

$$\begin{aligned}&\{\Phi^{3/4}(x)w(x)\}^{\nu} \times \sum_{k=1}^{n}\{w(x_{k,n})\frac{(1-|x_{k,n}|/a_{2n})^{1/2}}{(1-|x_{k,n}|/a_n)^{3/4}}\}^{-\nu}\sum_{s=0}^{l}|h_{skn}(\nu;x)| \\ &\leqslant C\log(1+n), \quad 0 \leqslant l \leqslant \nu-1. \end{aligned} \tag{1.2}$$

Definition 1.6. (1) Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$ and $j = 3, 4$. Let us assume that $Q \in C^{(j)}(\mathbb{R})$ and

$$\Big|\frac{Q^{(j-1)}(x)}{Q^{(j-2)}(x)}\Big| \sim \Big|\frac{Q^{(j-2)}(x)}{Q^{(j-3)}(x)}\Big|, \quad \Big|\frac{Q^{(j)}(x)}{Q^{(j-1)}(x)}\Big| \leq C\Big|\frac{Q^{(j-1)}(x)}{Q^{(j-2)}(x)}\Big| \tag{1.3}$$

hold for $|x| \geq K_1 > 0$, where K_1 is a constant, furthermore there exists $1 < \lambda < j/(j-1), j = 3.4$, such that

$$\frac{|Q'(x)|}{Q(x)^{\lambda}} \leq C, \quad |x| \geqslant K_2, \tag{1.4}$$

where K_2 is a positive constant. Then we write $w \in \mathcal{F}_{\lambda}(C^j+)$.
(2) Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let us define

$$\mu_+ := \limsup_{x\to\infty} \frac{Q''(x)}{Q'(x)} / \frac{Q'(x)}{Q(x)}, \mu_- := \liminf_{x\to\infty} \frac{Q''(x)}{Q'(x)} / \frac{Q'(x)}{Q(x)}.$$

If $\mu_+ = \mu_-$, then we say that the weight w is regular.
Remark 1.7. For Q in Example 1.2 we see that $w = \exp(-Q)$ are regular weights. If $Q \in \mathbf{C}^3(\mathbb{R})$ satisfies (1.3), then for the regular weights we have $w \in \mathcal{F}_{\lambda}(C^3+)$ (see [8, Corollary 5.5]).
Proposition 1.8. ([9, Appendix; Theorem A], cf. [8, Theorem 4.2]) Let $1 < \lambda < 3/2$ and $\mu, \alpha, \beta \in \mathbb{R}$. Then for $w = \exp(-Q) \in \mathcal{F}_{\lambda}(C^3+)$, we can construct a new weight $w_{\mu,\alpha,\beta} \in \mathcal{F}(C^2+)$ such that

$$(1+x^2)^{\mu}(1+Q(x))^{\alpha}(1+|Q'(x)|)^{\beta}w(x) \sim w_{\mu,\alpha,\beta}(x), x \in \mathbb{R}.$$

Now, let us define MRS-number for the weight $w_{\mu,\alpha,\beta} = \exp(-Q_{\mu;\alpha,\beta})$ by $a_n(Q_{\mu,\alpha,\beta})$, further we define the function T in Definition 1.1 (d) for the weight $w_{\mu,\alpha,\beta} = \exp(-Q_{\mu,\alpha,\beta})$ by $T_{\mu,\alpha,\beta}$. Then there exist $c, C > 0$ such that

$$a_{cn}(Q_{\mu,\alpha,\beta}) \leqslant a_n(Q) := a_n \leqslant a_{Cn}(Q_{\mu,\alpha,\beta})$$

and

$$T_{\mu,\alpha,\beta}(x) \sim T(x) x \in \mathbf{R}.$$

Let $w = \exp(-Q) \in \mathcal{F}_{\lambda}(C^3+),\ 1 < \lambda < 3/2$. By Proposition 1.8 we have

$$\begin{aligned}&w(x)\Phi(x)^{-3/4} \sim (1+x^2)^{3/8}(1+Q(x))^{-1/4}(1+|Q'(x)|)^{3/4}w(x) \\ &\sim W_0(x) := w_{\frac{3}{8},-\frac{1}{4},\frac{3}{4}}(x) \in \mathcal{F}(C^2+). \end{aligned} \tag{1.5}$$

And for $w = \exp(-Q) \in \mathcal{F}_{\lambda}(C^4+),\ 1 < \lambda < 4/3$ we have

$$T^{1/4}(x)\{\Phi(x)^{-3/4}w(x)\}^{\nu} \sim T^{1/4}(x)W_0^{\nu}(x) \sim W(x) \in \mathcal{F}(C^2+). \tag{1.6}$$

We can obtain the following theorem as an application of Theorem 1.5. For $f \in C(\mathbb{R})$, the degree of weighted polynomial approximation is defined by

$$E_n(w; f) := \inf_{P\in\mathcal{P}_n} \|w(f-P)\|_{L_\infty(\mathbb{R})}.$$

Theorem 1.9. Let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^4+)$, $1 < \lambda < 4/3$ be a regular weight. Then, for $f \in \mathbf{C}^\nu(\mathbb{R})$ with $|(1+|x|)^\nu T^{1/4}(x)(\Phi^{-3/4}w)^\nu(x)f^{(\nu)}(x)| \leqslant M$, $x \in \mathbb{R}$, where $M > 0$ is a constant, we have for some $0 < \alpha < 1$,

$$\|\{\Phi^{3/4}w\}^\nu(f - L_n(l,\nu,f))\|_{L_\infty(\mathbb{R})}$$
$$\leqslant C_\nu(\log(1+n))\{n^{-\alpha}E_{n-\nu}(T^{1/4}(\Phi^{-3/4}w)^\nu; f^{(\nu)}) + e(n)(\frac{a_n}{n})^{l+1}\}, \tag{1.7}$$

where $T^{1/4}(x)(\Phi^{-3/4}w)^\nu(x) \sim T^{1/4}(x)W_0^\nu(x) \sim W(x) \in \mathcal{F}(C^2+)$ and

$$e(n) = \begin{cases} 1, \, 0 \leqslant l \leqslant \nu - 2; \\ 0, \qquad l = \nu - 1. \end{cases}$$

Remark 1.10. We see $E_{n-\nu}(T^{1/4}(\Phi^{-3/4}w)^\nu; f^{(\nu)}) \to 0$ as $n \to \infty$.

2 Lemmas

Lemma 2.1. [3, Theorem 2.6 (2.2)] Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$. We have the following: For each $s = 0, 1, ..., \nu - 1$,

$$e_0(\nu,k,n) = 1, |e_{si}(\nu,k,n)| \leqslant C(\frac{n}{(a_{2n}^2 - x_{k,n}^2)^{1/2}})^{i-s}$$
$$s = 1, 2, \ldots, \nu - 1, \; i = s, s+1, ..., \nu - 1.$$

Lemma 2.2. [1, Theorem 2.7] Let $w \in \mathcal{F}(C^2+)$ and $\rho \geqslant 0$. Then uniformly for $n \geqslant 2$ and $1 \leqslant j \leqslant n$,

$$|(p_n w)(x)\Phi_n(x)^{1/4}| \leqslant C a_n^{-1/2}.$$

Lemma 2.3. [1, Theorem 2.5] Let $w \in \mathcal{F}(C^2+)$ and $\rho > -\frac{1}{2}$.
(a) There exists n_0 such that uniformly for $n \geqslant n_0$ and $1 \leqslant j \leqslant n$,

$$|(p_n' w)(x_{j,n})\Phi_n(x_{j,n})^{1/4}| \sim a_n^{-1/2}\varphi_n(x_{j,n})^{-1},$$

hence

$$|(p_n' w)(x_{j,n})\frac{1 - \frac{|x_{j,n}|}{a_{2n}}}{(1 - \frac{|x_{j,n}|}{a_n})^{1/4}}| \sim a_n^{-3/2}n.$$

(b)

$$\max_{x \in \mathbf{R}} |l_{j,n}(x)w(x)|w(x_{j,n})^{-1} \sim 1.$$

Lemma 2.4. (1) [6, Lemma 3.4 (3.18)] Uniformly for $t > 0$,

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}}.$$

(2) [6, Lemma 3.5 (3.27)-(3.29)] For $L > 1$,

$$a_{Lt} \sim a_t, \quad Q^{(j)}(a_{Lt}) \sim Q^{(j)}(a_t), \; j = 0, 1, \quad \text{and} \quad T(a_{Lt}) \sim T(a_t).$$

(3) [6, Lemma 3.6 (3.35)] For any fixed $L > 1$ and uniformly for $t > 0$,

$$1 - \frac{a_t}{a_{Lt}} \sim \frac{1}{T(a_t)}.$$

(4) If $|x| \leqslant a_{n/2}$, then uniformly we have

$$1-\frac{|x|}{a_{2n}} \sim 1-\frac{|x|}{a_n}.$$

Proof of (4). Let $|x| \leqslant a_{n/2}$.

$$1<\frac{1-\frac{|x|}{a_{2n}}}{1-\frac{|x|}{a_n}}=1+\frac{|x|}{a_n}\frac{1-\frac{a_n}{a_{2n}}}{1-\frac{|x|}{a_n}} \leqslant 1+\frac{a_{n/2}}{a_n}\frac{1-\frac{a_n}{a_{2n}}}{1-\frac{a_{n/2}}{a_n}} \leqslant C$$

by (2), (3) in this lemma. #

Lemma 2.5. [1, Theorem 2.2] Let $w(x)=\exp(-Q(x)) \in \mathcal{F}(C^2+)$. For the zeros $x_{j,n}$, we have the following.
(1) For $n \geqslant 1$ and $1 \leqslant j \leqslant n-1$,

$$x_{j,n}-x_{j+1,n} \sim \varphi_n(x_{j,n}), \quad \text{and} \quad \varphi_n(x_{j,n}) \sim \varphi_n(x_{j+1,n}).$$

(2) For the minimum positive zero $x_{[n/2],n}$ ($[n/2]$ is the largest integer $\leqslant n/2$), we have

$$x_{[n/2],n} \sim a_n n^{-1},$$

and for large enough n,

$$1-\frac{x_{1,n}}{a_n} \sim \delta_n.$$

Lemma 2.6. [6, Theorem 5.7 (b)] Let $M>0$. There exists $t_0>0$ such that for $t \geqslant t_0, x \in \mathbb{R}$, and

$$|y-x| \leqslant M\varphi_t(x),$$

we have

$$\varphi_t(x) \sim \varphi_t(y).$$

Lemma 2.7. Let $|x-x_{m,n}| \leqslant C\varphi_n(x_{m,n})$, and let $\Phi_n(x)$ be defined by (1.1).
(1) We see

$$\Phi_n(x) \sim (1-\frac{|x_{m,n}|}{a_n}). \tag{2.1}$$

(2) So we have

$$\Phi_n(x)^{\nu/2}(\frac{1}{\sqrt{1-\frac{|x_{m,n}|}{a_n}}})^i \leqslant C, \quad i=0,1,...,\nu-1. \tag{2.2}$$

Proof. (1) We see

$$\begin{aligned}\frac{\Phi_n(x)}{\Phi_n(x_{m,n})} &= \frac{1-\frac{|x|}{a_n}+\delta_n}{1-\frac{|x_{m,n}|}{a_n}+\delta_n} = 1+\frac{\frac{|x_{m,n}|}{a_n}-\frac{|x|}{a_n}}{1-\frac{|x_{m,n}|}{a_n}+\delta_n}\\
&\leqslant 1+\frac{C\frac{\varphi(x_{m,n})}{a_n}}{1-\frac{|x_{m,n}|}{a_n}+\delta_n} \leqslant 1+\frac{\frac{C}{n}\frac{1-\frac{|x_{m,n}|}{a_{2n}}}{\sqrt{1-\frac{|x_{m,n}|}{a_n}+\delta_n}}}{1-\frac{|x_{m,n}|}{a_n}+\delta_n}\\
&= 1+\frac{C}{n}\frac{1-\frac{|x_{m,n}|}{a_{2n}}}{(1-\frac{|x_{m,n}|}{a_n}+\delta_n)^{3/2}}\\
&\leqslant C\begin{cases}1+\frac{C}{n}\frac{1}{\sqrt{1-\frac{|x_{m,n}|}{a_n}}}, & |x_{m,n}|\leqslant a_{n/2};\\ 1+\frac{C}{n}\frac{nT(a_n)}{T(a_n)}, & a_{n/2}<|x_{m,n}|\end{cases}\\
&\leqslant C\begin{cases}1+\frac{C\sqrt{T(a_n)}}{n}, & |x_{m,n}|\leqslant a_{n/2};\\ 1+C, & a_{n/2}<|x_{m,n}|\end{cases}\\
&\leqslant C.\end{aligned}$$

Similarly, we have

$$\frac{\Phi_n(x_{m,n})}{\Phi_n(x)} \leqslant C.$$

In fact, since we see $\varphi_n(x) \sim \varphi_n(x_{m,n})$ by Lemma 2.6, we have the condition $|x-x_{m,n}| \leqslant C\varphi_n(x_{m,n})$. Then we can repeat the above consideration. Consequently, we have (2.1).
(2) By (2.1) we see

$$\Phi_n(x)^{\nu/2}\left(\frac{1-\frac{|x_{m,n}|}{a_{2n}}}{1-\frac{|x_{m,n}|}{a_n}+\delta_n}\right)^{i/2} \sim \Phi_n(x_{m,n})^{\nu/2}\left(\frac{1-\frac{|x_{m,n}|}{a_{2n}}}{1-\frac{|x_{m,n}|}{a_n}+\delta_n}\right)^{i/2} \leqslant C. \quad \#$$

Lemma 2.8. [7, Theorem 1 and Corollary 8] Let $w \in \mathcal{F}(C^2+)$. Let f be $s-1$ times continuously differentiable, and let $f^{(s-1)}(x)$ for some integer $s \geqslant 1$ be absolutely continuous in each compact interval. Let $wf^{(s)} \in L_\infty(\mathbb{R})$. Then we have

$$E_n(w;f) \leqslant C(\frac{a_n}{n})^s \|wf^{(s)}\|_{L_\infty(\mathbb{R})},$$

equivalently,

$$E_n(w;f) \leqslant C(\frac{a_n}{n})^s E_{n-s}(w;f^{(s)}).$$

Lemma 2.9. [4, Theorem 2.3] Let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^3+)$, and let it satisfy (2.3). Let $\nu \geqslant 1$ be an integer. We suppose that $f \in C^{(\nu-1)}(\mathbb{R})$ and $\lim_{|x|\to\infty} T^{1/4}(x)|f^{(\nu)}(x)|w(x) = 0$. Let

$$\|(f-P_{n,f,w})w\|_{L_\infty(\mathbb{R})} = E_n(w;f). \tag{2.3}$$

Then there exists an absolute constant $C_\nu > 0$, which depends only on ν such that, for $0 \leqslant j \leqslant \nu-1$ and $x \in \mathbb{R}$,

$$\begin{aligned}|(f^{(j)}(x) - P^{(j)}_{n,f,w}(x))w(x)| &\leqslant C_\nu T(x)^{j/2} E_{n-j}(w_{1/4};f^{(j)}) \qquad (2.4)\\
&\leqslant C_\nu T(x)^{j/2}(\frac{a_n}{n})^{\nu-j} E_{n-\nu}(w_{1/4};f^{(\nu)}),\end{aligned}$$

where $T^{1/4}w \sim w_{1/4} \in \mathcal{F}(C^2+)$.

3 Proofs of Theorems

We show Theorem 1.5. For each $x \in \mathbb{R}$ we define

$$|x - x_{m,n}| := \min\{|x - x_{j,n}|,\ j = 0, 1, 2, ..., n+1\}, \tag{3.1}$$

where

$$x_{0,n} := \frac{a_n + x_{1,n}}{2}, \quad x_{n+1,n} := -x_{0,n}.$$

Proof of Theorem 1.5. (Case 1) For $x_{m,n}$ in (3.1) we see

$$\begin{aligned}
&\{\Phi^{3/4}(x)w(x)\}^{\nu}|h_{mn}(x)|\{w(x_{m,n})\frac{(1-|x_{m,n}|/a_{2n})^{1/2}}{(1-|x_{m,n}|/a_n)^{3/4}}\}^{-\nu} \\
&\leqslant \Phi^{3\nu/4}(x)\{|l_{m,n}(x)|w(x)w(x_{m,n})^{-1}\}^{\nu} \times \{\frac{(1-|x_{m,n}|/a_n)^{3/4}}{(1-|x_{m,n}|/a_{2n})^{1/2}}\}^{\nu}\sum_{i=0}^{\nu-1}|e_i(\nu,m,n)||x-x_{m,n}|^i \\
&\leqslant C\Phi^{3\nu/4}(x)\{\frac{(1-|x_{m,n}|/a_n)^{3/4}}{(1-|x_{m,n}|/a_{2n})^{1/2}}\}^{\nu}\sum_{i=0}^{\nu-1}|e_i(\nu,m,n)||x-x_{m,n}|^i \\
&\text{by Lemma 2.3 (b)} \\
&\leqslant C\Phi^{3\nu/4}(x)\{\frac{(1-|x_{m,n}|/a_n)^{3/4}}{(1-|x_{m,n}|/a_{2n})^{1/2}}\}^{\nu}\sum_{i=0}^{\nu-1}(\frac{n}{\sqrt{a_{2n}^2-x_{m,n}^2}})^i\varphi_n(x_{m,n})^i \\
&\text{by Lemma 2.1} \\
&\leqslant C\Phi^{3\nu/4}(x)\{\frac{(1-|x_{m,n}|/a_n)^{3/4}}{(1-|x_{m,n}|/a_{2n})^{1/2}}\}^{\nu}\sum_{i=0}^{\nu-1}(\frac{1}{1-\frac{|x_{m,n}|}{a_{2n}}}\frac{1-\frac{|x_{m,n}|}{a_{2n}}}{\sqrt{1-\frac{|x_{m,n}|}{a_n}}})^i \\
&= C\Phi^{3\nu/4}(x)\frac{(1-|x_{m,n}|/a_n)^{3\nu/4}}{(1-|x_{m,n}|/a_{2n})^{\nu/2}}\sum_{i=0}^{\nu-1}(\frac{1}{\sqrt{1-\frac{|x_{m,n}|}{a_n}}})^i \\
&\leqslant C\Phi^{3\nu/4}(x)(1-|x_{m,n}|/a_n)^{-\nu/4}\sum_{i=0}^{\nu-1}(1-|x_{m,n}|/a_n)^{\nu/2}(\frac{1}{\sqrt{1-\frac{|x_{m,n}|}{a_n}}})^i \leqslant C
\end{aligned}$$

by Lemma 2.7 (2). Next, we estimate

$$\sum_{k\neq m} := \{\Phi^{3/4}(x)w(x)\}^{\nu}\sum_{k=1,k\neq m}^{n}|h_{kn}(x)|\{w(x_{k,n})\frac{(1-|x_{k,n}|/a_{2n})^{1/2}}{(1-|x_{k,n}|/a_n)^{3/4}}\}^{-\nu}. \tag{3.2}$$

(Case 2) For $|x| < x_{0,n}$ we will estimate $\sum_1$ in (3.2). Noting Lemma 2.1,

$$\begin{aligned}
\sum_1 \leqslant &\sum_{k=1,k\neq m}^{n}\{\Phi(x)^{3/4}w(x)\}^{\nu}\{w(x_{k,n})\frac{(1-|x_{k,n}|/a_{2n})^{1/2}}{(1-|x_{k,n}|/a_n)^{3/4}}\}^{-\nu} \\
&\times|l_{k,n}(x)|^{\nu}\sum_{i=0}^{\nu-1}(\frac{n}{\sqrt{a_{2n}^2-x_{k,n}^2}})^i|x-x_{k,n}|^i
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&= \sum_{k=1,k\neq m}^{n} \Phi(x)^{\nu/2} \Big| \frac{p_n(x)w(x)\Phi(x)^{1/4}}{|x-x_{k,n}|(p_n'w)(x_{k,n})\frac{1-\frac{|x_{k,n}|}{a_{2n}}}{(1-\frac{|x_{k,n}|}{a_n})^{1/4}}} \Big|^{\nu} \\
&\quad \times \{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \sum_{i=0}^{\nu-1} (\frac{n}{\sqrt{a_{2n}^2-x_{k,n}^2}})^i |x-x_{k,n}|^i \\
&\leqslant C \sum_{k=1,k\neq m}^{n} (\frac{a_n}{n})^{\nu} \Phi(x)^{\nu/2} \{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \\
&\quad \times \sum_{i=0}^{\nu-1} (\frac{n}{a_n})^i (\frac{1}{1-\frac{|x_{k,n}|}{a_{2n}}})^{i/2} |x-x_{k,n}|^{i-\nu} \\
&\quad \text{by Lemma 2.2, 2.3(a)} \\
&\leqslant C \sum_{k=1,k\neq m}^{n} (\frac{a_n}{n})^{\nu} \Phi(x)^{\nu/2} \{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \\
&\quad \times \sum_{i=0}^{\nu-1} (\frac{n}{a_n})^i (\frac{1}{1-\frac{|x_{k,n}|}{a_{2n}}})^{i/2} (\frac{1}{\sum_{j=m}^{k} \varphi_n(x_{j,n})})^{\nu-i} \\
&\quad \text{by Lemma 2.5 (1)} \\
&\leqslant C \sum_{k=1,k\neq m}^{n} (\frac{a_n}{n})^{\nu} \Phi(x)^{\nu/2} \{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \\
&\quad \times \sum_{i=0}^{\nu-1} (\frac{n}{a_n})^{\nu} (\frac{1}{1-\frac{|x_{k,n}|}{a_{2n}}})^{i/2} (\frac{1}{\sum_{j=m}^{k} \frac{1-|x_{j,n}|/a_{2n}}{\sqrt{1-|x_{j,n}|/a_n}}})^{\nu-i} \\
&\leqslant C \sum_{k=1,k\neq m}^{n} \Phi(x)^{\nu/2} \{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \\
&\quad \times \sum_{i=0}^{\nu-1} (\frac{1}{1-\frac{|x_{k,n}|}{a_{2n}}})^{i/2} (\frac{1}{\sum_{j=m}^{k} \sqrt{1-\frac{|x_{j,n}|}{a_n}}})^{\nu-i} \\
&\leqslant C \sum_{k=1,k\neq m}^{n} \Phi(x)^{\nu/2} (1-\frac{|x_{k,n}|}{a_n})^{\nu/2} \\
&\quad \times \sum_{i=0}^{\nu-1} (1-\frac{|x_{k,n}|}{a_{2n}})^{(\nu-i)/2} (\frac{1}{|k-m|\min\{\Phi_n(x)^{1/2},\Phi_n(x_{k,n})^{1/2}\}})^{\nu-i} \\
&\leqslant C \sum_{k=1,k\neq m}^{n} \sum_{i=0}^{\nu-1} \Phi(x)^{\nu/2} (1-\frac{|x_{k,n}|}{a_n})^{\nu/2} (\frac{1}{|k-m|\min\{\Phi_n(x)^{1/2},\Phi_n(x_{k,n})^{1/2}\}})^{\nu-i} \\
&\leqslant C \log n.
\end{aligned}
$$

(Case 3) For $x_{0,n} \leqslant |x|$ we will estimate $\sum_2$ in (3.2). We start from (3.3).

$$\begin{aligned}
\sum_2 &\leqslant C\sum_{k=1}^{n}(\frac{a_n}{n})^{\nu}\Phi(x)^{\nu/2}\{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \\
&\quad\times\sum_{i=0}^{\nu-1}(\frac{n}{a_n})^{i}(\frac{1}{1-\frac{|x_{k,n}|}{a_{2n}}})^{i/2}|x-x_{k,n}|^{i-\nu} \\
&\leqslant C\sum_{k=1}^{n}(\frac{a_n}{n})^{\nu}\Phi(x)^{\nu/2}\{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \\
&\quad\times\sum_{i=0}^{\nu-1}(\frac{n}{a_n})^{i}(\frac{1}{1-\frac{|x_{k,n}|}{a_{2n}}})^{i/2}|\pm x_{0,n}-x_{k,n}|^{i-\nu} \\
&\leqslant C\sum_{k=1}^{n}(\frac{a_n}{n})^{\nu}\Phi(x)^{\nu/2}\{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \\
&\quad\times\sum_{i=0}^{\nu-1}(\frac{n}{a_n})^{i}(\frac{1}{1-\frac{|x_{k,n}|}{a_{2n}}})^{i/2}(\frac{1}{\sum_{j=0}^{k}\varphi_n(x_{j,n})})^{\nu-i} \\
&\quad\text{by Lemma 2.5 (1)} \\
&\leqslant C\sum_{k=1}^{n}(\frac{a_n}{n})^{\nu}\Phi(x)^{\nu/2}\{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \\
&\quad\times\sum_{i=0}^{\nu-1}(\frac{n}{a_n})^{\nu}(\frac{1}{1-\frac{|x_{k,n}|}{a_{2n}}})^{i/2}(\frac{1}{\sum_{j=0}^{k}\frac{1-|x_{k,n}|/a_{2n}}{\sqrt{1-|x_{k,n}|/a_n}}})^{\nu-i} \\
&\leqslant C\sum_{k=1}^{n}\Phi(x)^{\nu/2}\{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \\
&\quad\times\sum_{i=0}^{\nu-1}(\frac{1}{1-\frac{|x_{k,n}|}{a_{2n}}})^{i/2}(\frac{1}{\sum_{j=0}^{k}\sqrt{1-\frac{|x_{j,n}|}{a_n}}})^{\nu-i} \\
&\leqslant C\sum_{k=1}^{n}\Phi(x)^{\nu/2}\{(1-\frac{|x_{k,n}|}{a_n})(1-\frac{|x_{k,n}|}{a_{2n}})\}^{\nu/2} \\
&\quad\times\sum_{i=0}^{\nu-1}(\frac{1}{1-\frac{|x_{k,n}|}{a_{2n}}})^{i/2}(\frac{1}{k\Phi_n(x_{0,n})^{1/2}})^{\nu-i} \\
&\leqslant C\sum_{k=1}^{n}(1-\frac{|x_{k,n}|}{a_n})^{\nu/2}\sum_{i=0}^{\nu-1}(1-\frac{|x_{k,n}|}{a_{2n}})^{(\nu-1)/2}\Phi(x)^{\nu/2}(\frac{1}{k\Phi_n(x_{0,n})^{1/2}})^{\nu-i} \\
&\leqslant C\log n \quad \text{by } \Phi_n(x)\sim\Phi_n(x_{0,n}).
\end{aligned}$$

(Case 4) Finally, for each $s \geqslant 1$ we estimate

$$\sum_s := \{\Phi^{3/4}(x)w(x)\}^{\nu}\sum_{k=1}^{n}|h_{skn}(x)|\{w(x_{k,n})\frac{(1-|x_{k,n}|/a_{2n})^{1/2}}{(1-|x_{k,n}|/a_n)^{3/4}}\}^{-\nu}.$$

We may estimate

$$\begin{aligned}
\sum_s{}' := &\ \{\Phi^{3/4}(x)w(x)\}^{\nu}\sum_{k=1}^{n}\{w(x_{k,n})\frac{(1-|x_{k,n}|/a_{2n})^{1/2}}{(1-|x_{k,n}|/a_n)^{3/4}}\}^{-\nu} \\
&\times|l_{k,n}(x)|^{\nu}\sum_{i=s}^{\nu-1}(\frac{n}{\sqrt{a_{2n}^2-x_{k,n}^2}})^{i-s}|x-x_{k,n}|^{i}.
\end{aligned}$$

It is, however, easy. In fact, from the above estimation we have

$$\begin{aligned}\sum_s{}' &\leqslant \{\Phi^{3/4}(x)w(x)\}^{\nu}(\frac{a_n}{n})^s\sum_{k=1}^{n}\{w(x_{k,n})\frac{(1-|x_{k,n}|/a_{2n})^{1/2}}{(1-|x_{k,n}|/a_n)^{3/4}}\}^{-\nu}\\ &\times|l_{k,n}(x)|^{\nu}\sum_{i=s}^{\nu-1}(\frac{n}{\sqrt{a_{2n}^2-x_{k,n}^2}})^i|x-x_{k,n}|^i\\ &\leqslant C(\frac{a_n}{n})^s\log(1+n). \end{aligned}\tag{3.4}$$

Consequently, the proof of Theorem 1.5 is complete. #

Lemma 3.1. Let $\|(1+|x|)^{\nu}W_0^{\nu}(x)f^{(\nu)}(x)\|_{L_\infty(\mathbb{R})}\leqslant M$. Then we have

$$\|f^{(s)}(x)W_0^{\nu}(x)\|_{L_\infty(\mathbb{R})}\leqslant M_s, s=0,1,...,\nu,$$

where $M, M_s>0$ are constants.
Proof. We may suppose $x\geqslant 0$. Since $(1+x)^{\nu}W_0^{\nu}(x)$ is quasi-decreasing, we see

$$\begin{aligned}&|(1+x)^{\nu-1}W_0^{\nu}(x)f^{(\nu-1)}(x)|=|(1+x)^{\nu-1}W_0^{\nu}(x)\{\int_0^x f^{(\nu)}(t)dt+f^{(\nu-1)}(0)\}|\\ &\leqslant C\int_0^x(1+t)^{\nu}W_0^{\nu}(t)|f^{(\nu)}(t)|dt\frac{1}{1+x}+|(1+x)^{\nu-1}W_0^{\nu}(x)f^{(\nu-1)}(0)|\\ &\leqslant M\int_0^x dt\frac{1}{1+x}+(1+x)^{\nu-1}W_0^{\nu}(x)|f^{(\nu-1)}(0)|.\end{aligned}$$

So we have

$$\|(1+|x|)^{\nu-1}W_0^{\nu}(x)f^{(\nu-1)}(x)\|_{L_\infty(\mathbb{R})}\leqslant M_{\nu-1}.$$

Therefore, inductively, we have

$$\|(1+|x|)^{\nu-s}W_0^{\nu}(x)f^{(\nu-s)}(x)\|_{L_\infty(\mathbb{R})}\leqslant M_{\nu-s},\quad s=0,1,...,\nu.$$

Consequently, we obtain the result. #

Proof of Theorem 1.9. Let $P_{n,f,T^{(\nu-1)/2}W_0^{\nu}}\in\mathcal{P}_n$ be the best approximation of f with respect to the weight W_0^{ν}, that is,

$$\|W_0^{\nu}(f-P_{n,f,W_0^{\nu}})\|_{L_\infty(\mathbb{R})}=E_n(W_0^{\nu};f).$$

We see

$$\begin{aligned}f(x)-L_n(l,\nu,f;x)=&\ f(x)-P_{n,f,W_0^{\nu}}(x)-L_n(\nu-1,\nu,f-P_{n,f,W_0^{\nu}};x)\\ &+\sum_{j=1}^{n}\sum_{s=l+1}^{\nu-1}f^{(s)}(x_{j,n})h_{sjn}(l,\nu;x),\end{aligned}$$

where if $l=\nu-1$, then the last term is equal to 0. We see that

$$\begin{aligned}&(\Phi^{3/4}(x)w(x))^{\nu}|f(x)-P_{n,f,W_0^{\nu}}(x)|\leqslant C\|W_0^{\nu}(f(x)-P_{n,f,W_0^{\nu}}(x))\|_{L_\infty(\mathbb{R})}\\ &=CE_n(W_0^{\nu};f)\leqslant C(\frac{a_n}{n})^{\nu}E_{n-\nu}(W_0^{\nu};f^{(\nu)})\end{aligned}\tag{3.5}$$

by Lemma 2.8. We estimate $|L_n(\nu-1,\nu,f-P_{n,f,W_0^\nu};x)|$.

$$\begin{aligned}
&(\Phi^{3/4}(x)w(x))^\nu|L_n(\nu-1,\nu,f-P_{n,f,W_0^\nu};x)| \\
=\ &(\Phi^{3/4}(x)w(x))^\nu|\sum_{k=1}^{n}\sum_{s=0}^{\nu-1}(f^{(s)}(x_{k,n})-P^{(s)}_{n,f,W_0^\nu}(x_{k,n}))h_{skn}(\nu;x)| \\
\leqslant\ &(\Phi^{3/4}(x)w(x))^\nu\times\sum_{k=1}^{n}\sum_{s=0}^{\nu-1}|f^{(s)}(x_{k,n})-P^{(s)}_{n,f,W_0^\nu}(x_{k,n})|\{w(x_{k,n})\Phi^{-3/4}(x_{k,n})\}^\nu \\
&\times|h_{skn}(\nu;x)|\{w(x_{k,n})\Phi^{-3/4}(x_{k,n})\}^{-\nu}. \qquad (3.6)
\end{aligned}$$

Now, we see by Lemma 2.9 with the weight $W_0^\nu(x)$,

$$\begin{aligned}
&|f^{(s)}(x_{k,n})-P^{(s)}_{n,f,W_0^\nu}(x_{k,n})|\{w(x_{k,n})\Phi^{-3/4}(x_{k,n})\}^\nu \\
\leqslant\ &C|f^{(s)}(x_{k,n})-P^{(s)}_{n,f,W_0^\nu}(x_{k,n})|W_0^\nu(x_{k,n}) \\
\leqslant\ &CT_{W_0^\nu}(a_n)^{s/2}E_{n-s}(T^{1/4}_{W_0^\nu}W_0^\nu;f^{(s)}) \\
\leqslant\ &CT_{W_0^\nu}(a_n)^{s/2}(\frac{a_{n,W_0^\nu}}{n})^{\nu-s}E_{n-\nu}(T^{1/4}_{W_0^\nu}W_0^\nu;f^{(\nu)}), \qquad (3.7)
\end{aligned}$$

where $T_{W_0^\nu},\ a_{n,W_0^\nu}$ correspond to the weight $W_0^\nu(x)$. Since w is regular, we have for any $\eta>0$

$$T_{W_0^\nu}(x)\sim T_{W^\nu}(x)\sim T_w(x)=T(x)\leqslant C_\eta n^\eta,$$

where $C_\eta>0$ is a constant depending only on η (see [8]). Furthermore, we see that for some $0<\delta<1$

$$a_{n,W_0^\nu}\sim a_{n/\nu,W_0}\sim a_{n/\nu,w}\sim a_{n/\nu}\leqslant Cn^\delta.$$

So we have

$$\begin{aligned}
&T_{W_0}(a_n)^{s/2}(\frac{a_{n,W_0}}{n})^{\nu-s}\leqslant T(a_n)^{\nu/2}(\frac{a_n}{n})\leqslant CC_\eta n^{-(1-\eta-\delta)} \qquad (3.8)\\
=\ &CC_\eta n^{-\frac{1-\delta}{2}}=:CC_\eta n^{-\alpha}\quad(0<\alpha<1)
\end{aligned}$$

with $\eta=(1-\delta)/2$. Therefore, (3.7) means

$$|f^{(s)}(x_{k,n})-P^{(s)}_{n,f,W_0^\nu}(x_{k,n})|\{w(x_{k,n})\Phi^{-3/4}(x_{k,n})\}^\nu\leqslant CC_\eta n^{-\alpha}E_{n-\nu}(T^{1/4}W_0^\nu;f^{(\nu)}).$$

Hence, from (3.7) and Theorem 1.5 we have

$$\begin{aligned}
&(\Phi^{3/4}(x)w(x))^\nu|L_n(\nu-1,\nu,f-P_{n,f,W_0^\nu};x)| \\
\leqslant\ &Cn^{-\alpha}E_{n-\nu}(T^{1/4}_{W_0^\nu}W_0^\nu;f^{(\nu)}) \\
&\times(\Phi^{3/4}(x)w(x))^\nu\sum_{k=1}^{n}\{w(x_{k,n})\frac{(1-|x_{k,n}|/a_{2n})^{1/2}}{(1-|x_{k,n}|/a_n)^{3/4}}\}^{-\nu}\sum_{s=0}^{\nu-1}|h_{skn}(\nu;x)| \\
\leqslant\ &Cn^{-\alpha}\log(1+n)E_{n-\nu}(T^{1/4}W_0^\nu;f^{(\nu)}). \qquad (3.9)
\end{aligned}$$

Here we note $\Phi(x)^{3/4}\leqslant C\frac{(1-|x|/a_n)^{3/4}}{(1-|x|/a_{2n})^{1/2}}$. Finally, for $l\leqslant\nu-2$ we estimate

$$\sum:=(\Phi^{3/4}(x)w(x))^\nu\sum_{k=1}^{n}\sum_{s=l+1}^{\nu-1}f^{(s)}(x_{k,n})h_{skn}(\nu;x).$$

We see

$$\begin{aligned}
|\sum|\leqslant\ &(\Phi^{3/4}(x)w(x))^\nu\sum_{k=1}^{n}\sum_{s=l+1}^{\nu-1}|f^{(s)}(x_{k,n})|\{w(x_{k,n})\Phi(x_{k,n})^{-3/4}\}^\nu \\
&\times|h_{skn}(\nu;x)|\{w(x_{k,n})\frac{(1-|x_{k,n}|/a_{2n})^{1/2}}{(1-|x_{k,n}|/a_n)^{3/4}}\}^{-\nu}.
\end{aligned}$$

Here, noting (1.5) and Lemma 3.1, we have

$$|f^{(s)}(x_{k,n})|\{w(x_{k,n})\Phi(x_{k,n})^{-3/4}\}^{\nu} \leqslant M_s.$$

Now, using (3.4), we have

$$\begin{aligned} |\sum| &\leqslant (\Phi^{3/4}(x)w(x))^{\nu} \times \sum_{k=1}^{n}\sum_{s=l+1}^{\nu-1}|h_{skn}(\nu;x)|\{w(x_{k,n})\frac{(1-|x_{k,n}|/a_{2n})^{1/2}}{(1-|x_{k,n}|/a_n)^{3/4}}\}^{-\nu} \\ &\leqslant (\frac{a_n}{n})^{l+1}\log(1+n). \end{aligned} \quad (3.10)$$

From (3.5), (3.8) and (3.9) we have the result. #

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