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On the Existence and Uniqueness of the Solution of Pollutants Transport Problem in a River

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Abstract In this paper, we examine a problem of pollutant transport described by a nonlinear parabolic Partial differential equation (PDE) on a planar domain with obstacles. We then establish an existence and uniqueness result for this corresponding problem with Neumann.boundary conditions.

Keywords: Nonlinear parabolic partial differential equation, pollutant transport, planar domain

1 Introduction

The propagation of pollutants in a river obeys to the physical laws of transport which consider that the concentration $c(t,x)$ at a point x at time t measured in term of oxygen demand satisfies the following partial differential equation

$$\frac{\partial c}{\partial t} - \nabla.(\lambda \nabla c) + \mathbf{V}.\nabla c + F(c) = f \quad \text{in} \,]0,T[\times \Omega \tag{1}$$

where Ω is a $\mathbb{R}^2$ bounded domain which represents the surface of a portion of a river. The domain Ω is supposed to be of a complex geometrical form that can contain obstacles and we shall assume that the study domain satisfies the Lipschitz boundary condition. In the above equation, the term λ designs the pollutant dispersion coefficient, $\mathbf{V}$ is the velocity field of the fluid in the river, $F(c)$ is a non-linear term that describes the phenomen on of chemical reaction due to the presence of pollutants in the river and, f is the source term.

In addition to equation (1), following Neuman boundary and initial conditions are considered to complete the description of the problem:

$$\frac{\partial c}{\partial n} = g \text{ on } \,]0,T[\times \Gamma_1 \cup \Gamma_3 \tag{2}$$

$$\frac{\partial c}{\partial n} = 0 \text{ on } \,]0,T[\times \Gamma_2 \cup \Gamma_4 \cup \Gamma_5 \tag{3}$$

$$c(0,x) = c_0(x) \quad x \in \Omega \tag{4}$$

where the boundary of Ω is split into Γ_i, with $i = 1,2,3,4,5$ (see figure 1) and where g and c_0 are given functions. Here g represents pollutants concentration flow between two portions of the boundary Γ_1 and Γ_3, while c_0 is the initial concentration in all the domain Ω.

This is a diffusion-reaction -convection problem that belongs to the class of nonlinear parabolic Partial differential equations system. For these such problems, many papers have already been published (see for example [1,**?**]). The most frequently seen approaches start out by seeking some form of weak solution as a limit in some large space with the equation interpreted in a quite generalized sence (weak solution) and then look for regularity results to hope these are solutions in something closer to a classical sence. For the existence and uniqueness of these types of problems, two approaches seem to stand out. The first ones is based on the use of semi-group theory which is mainly interested in finding strong solutions while the second approach uses the fixed point theory via a weak formulation of the problem. In this paper, we state and demonstrate the existence and uniqueness of a weak solution of our problem , using an approach quite similar to that already used in [2] for the study of lakes sedimentation problem.

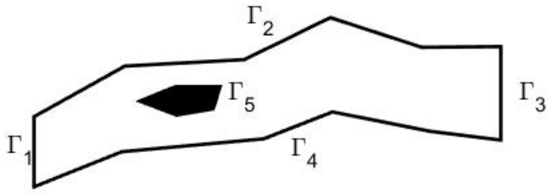

Figure 1. An illustration of the boundary of the study domain representing the surface of a portion of river with obstacles in the form of islets.

2 Main Result

Suppose that the solution of the system (1) - (4) is sufficiently regular. Then, multiplying equations (1) and (4) by $\phi \in H^1(\Omega)$ and, integrating over the domain, we finally obtain following weak equations

$$\frac{d}{dt}\int_\Omega c\phi + \int_\Omega \lambda\nabla c.\nabla\phi + \int_\Omega G(c)\phi = \int_\Omega f\phi + \int_{\Gamma_1\cup\Gamma_3} \lambda g\phi d\Gamma \tag{5}$$

$$\int_\Omega c(0,.)\phi = \int_\Omega c_0\phi \tag{6}$$

$\forall\phi \in H^1(\Omega)$. In (5) $d\Gamma$ denotes the surface measurement and we have set

$$G(c) = \mathbf{V}.\nabla c + F(c). \tag{7}$$

In the following, a function c is said to be a weak solution of the problem (1)-(4) if it satisfies (5)-(6). Furthermore, we will assume following hypotheses.

Hypothesis 1 *Functions g, c_0 and F are assumed to be differentiable with respect to each of their arguments.*

Hypothesis 2 *Function G satisfies growth condition*

$$(G(u) - G(v))\,(u - v) \geq 0, \quad \forall u,\ v \in \mathbb{R} \tag{8}$$

and

$$G(0) = 0. \tag{9}$$

Our main result is the following.

Theorem 1 *According to hypotheses 1 and 2, there exists $c \in C^0(0, T; H^1(\Omega))$ a unique weak solution of* (1)-(4) *that satisfies the weak formulation* (5)-(6).

3 Proof of the Theorem 1

We prove this theorem in three stages.

3.1 Step 1

$H^1(\Omega)$ being a separable Hilbert space, it admits a Hilbert basis $\{\varphi_i\}$ that satisfies

$$\int_\Omega \varphi_i \varphi_j = \delta_{ij} \tag{10}$$

For fixed k, , we set $\mathcal{V}_k = span\{\varphi_1, \varphi_2, ..., \varphi_k\}$ a finite dimension subspace of $H^1(\Omega)$ and we look for $c_k \in C^0(0,T;\mathcal{V}_k)$ that satisfies

$$\frac{d}{dt}\int_\Omega c_k\varphi_i + \int_\Omega \lambda\nabla c_k.\nabla\varphi_i + \int_\Omega G(c_k)\varphi_i = \int_\Omega f\varphi_i + \int_{\Gamma_1\cup\Gamma_3} \lambda g\varphi_i d\Gamma \tag{11}$$

$$\int_\Omega c_k(0)\varphi_i = \int_\Omega c_0\varphi_i \tag{12}$$

$\forall \;\; i = 1, ..., k.$
Searching solutions of (11)-(12) leads to determine coefficients $c_{kj}(t)$ such as $c_k(t,.) = \sum_{j=1}^{k} c_{kj}(t)\varphi_j$. Replacing this expression in equations (11) and (12) yields the following Cauchy system

$$\begin{aligned}\frac{d}{dt}c_{ki}(t) = &-\sum_{j=1}^{k} c_{kj}(t)\int_\Omega \lambda\nabla\varphi_j.\nabla\varphi_i - \int_\Omega G\left(\sum_{j=1}^{k} c_{kj}(t)\varphi_j\right)\varphi_i \\ &+\int_\Omega f\varphi_i + \int_{\Gamma_1\cup\Gamma_3}\lambda g\varphi_i d\Gamma, \quad i = 1,...,k\end{aligned} \tag{13}$$

and

$$c_{ki}(0) = \int_\Omega c_0\varphi_i, \quad i = 1, ..., k. \tag{14}$$

Thanks to hypotheses 1 and 2, the second member of the equation (13) is differentiable with respect to each of its arguments. By Cauchy-Lipschitz theorem, it follows that the system (13)-(14) admits a unique solution. We have therefore shown that there is a unique solution $c_k \in C^0(0,T;\mathcal{V}_k)$ satisfying (11)-(12).

3.2 Step 2

Having established the existence of a sequence of functions $c_k \in C^0(0,T;\mathcal{V}_k)$ (11)-(12) we try to establish here some a priori estimates.

An a priori estimate of ∇c_k Multiplying each equation of (13)-(14) by $c_{kj}(t)$, then summing over j, we obtain for all t

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (c_k)^2 + \int_\Omega \lambda|\nabla c_k|^2 + \int_\Omega G(c_k)c_k = \int_\Omega fc_k + \int_{\Gamma_1\cup\Gamma_3}\lambda g c_k d\Gamma \tag{15}$$

Thanks to hypotheses 1 and 2, ones obtains

$$\frac{d}{dt}\int_\Omega |c_k(t)|^2 + \int_\Omega |\nabla c_k(t)|^2 \leq \sigma\left(\int_\Omega |f||c_k(t)| + \int_{\partial\Omega}\lambda|g||c_k(t)|d\Gamma\right) \tag{16}$$

where σ is a positive constant. By successively applying the trace theorem, Cauchy-Schwartz inequality and Young inequality we obtain

$$\frac{d}{dt}\|c_k(t)\|^2_{L^2(\Omega)} + \|\nabla c_k(t)\|^2 \leq \sigma\left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Gamma)} + \|c_k(t)\|^2_{L^2(\Omega)}\right) \tag{17}$$

Integrating over $[0,\; T]$ we obtain

$$\begin{aligned}\|c_k(T)\|^2_{L^2(\Omega)} + \int_0^T \|\nabla c_k(t)\|^2_{L^2(\Omega)} \leq \|c_{k0}\|^2_{L^2(\Omega)} + \sigma\Bigg(&\int_0^T \|f\|^2_{L^2(\Omega)} \\ &+ \int_0^T \|g\|^2_{L^2(\Gamma)} + \int_0^T \|c_k(t)\|^2_{L^2(\Omega)}\Bigg)\end{aligned}$$

Knowing that $\|c_k(T)\|^2_{L^2(\Omega)} \leq \int_0^T \|c_k(t)\|^2_{L^2(\Omega)}$ and $\|c_{k0}\|^2_{L^2(\Omega)} \leq \|c_0\|^2_{L^2(\Omega)}$ we finally deduce

$$\|\nabla c_k(t)\|_{C^0(0,T;L^2(\Omega))} \leq \kappa \left(\|c_0\|_{L^2(\Omega)} + \|f\|_{C^0(0,T;L^2(\Omega))} + \|g\|_{C^0(0,T;L^2(\Gamma))}\right) \tag{18}$$

where κ is a positive constant.

An a priori estimate of c_k Considering again the inequality (16) ones can deduce

$$\frac{d}{dt}\|c_k(t)\|^2_{L^2(\Omega)} \leq \sigma \left(\int_\Omega |f||c_k(t)| + \int_{\partial\Omega} \lambda |g||c_k(t)| d\Gamma\right)$$

and, thanks to Cauchy-Schwartz inequality we obtain

$$\frac{d}{dt}\|c_k(t)\|_{L^2(\Omega)} \leq \sigma \left(\int_\Omega |f| + \int_{\partial\Omega} \lambda |g| d\Gamma\right).$$

Thus, integrating over $[0,t]$ for all $t \leq T$, its follows

$$\|c_k(t)\|_{L^2(\Omega)} \leq \kappa \left(\|c_0\|_{L^2(\Omega)} + \int_0^T \|f\|_{L^2(\Omega)} + \int_0^T \|g\|_{L^2(\Gamma)}\right)$$

for a positive constant κ. This last inequality finally implies

$$\|c_k\|_{C^0(0,T;L^2(\Omega))} \leq \kappa \left(\|c_0\|_{L^2(\Omega)} + \int_0^T \|f\|_{L^2(\Omega)} + \int_0^T \|g\|_{L^2(\Gamma)}\right). \tag{19}$$

3.3 Step 3

The existence According to the a priori inequalities (18) and (19) the sequence (c_k) is bounded in $C^0\left(0,T;H^1(\Omega)\right)$, then it follows that it admits a subsequence also denoted by (c_k) such that

$$c_k \rightharpoonup c \text{ weakly in } C^0\left(0,T;H^1(\Omega)\right) \tag{20}$$

and by the compact injection property

$$c_k \longrightarrow c \text{ strongly in } C^0\left(0,T;L^2(\Omega)\right). \tag{21}$$

Now consider a function $\phi \in H^1(\Omega)$. As (φ_i) is an Hilbert basis of $H^1(\Omega)$ then there exists a sequence of reals (α_i) such that

$$v_k = \sum_{i=1}^k \alpha_i \varphi_i \longrightarrow \phi \text{ in } H^1(\Omega). \tag{22}$$

On the other hand, from (11) and (12) we also have

$$\frac{d}{dt}\int_\Omega c_k v_k + \int_\Omega \lambda \nabla c_k . \nabla v_k + \int_\Omega G(c_k) v_k = \int_\Omega f v_k + \int_{\Gamma_1 \cup \Gamma_3} \lambda g v_k d\Gamma \tag{23}$$

$$\int_\Omega c_k(0) v_k = \int_\Omega c_0 v_k. \tag{24}$$

Then, thanks to (20), (21), (22) and passing to the limit in these above expressions, ones finally obtains

$$\frac{d}{dt}\int_\Omega c\phi + \int_\Omega \lambda \nabla c . \nabla \phi + \int_\Omega G(c)\phi = \int_\Omega f\phi + \int_{\Gamma_1 \cup \Gamma_3} \lambda g \phi d\Gamma \tag{25}$$

$$\int_\Omega c(0)\phi = \int_\Omega c_0 \phi \tag{26}$$

for all $\phi \in H^1(\Omega)$. This establishes the existence of a weak solution of the problem (1)-(3) in $C^0\left(0,T;H^1(\Omega)\right)$.

3.4 The Uniqueness

Suppose there exists two functions $u(t)$ and $v(t)$ weak solutions of the problem (1)-(3). Then replacing these functions in equation (5) we obtain

$$\frac{d}{dt}\int_{\Omega}(u(t)-v(t))\phi+\int_{\Omega}\lambda\nabla(u(t)-v(t)).\nabla\phi+\int_{\Omega}G(u(t)-v(t))\phi=0$$

for all $\phi\in H^1(\Omega)$. By setting $\phi=u-v$ this equation becomes

$$\frac{d}{dt}\|u-v\|^2_{L^2(\Omega)}+\int_{\Omega}\lambda|\nabla(u-v)|^2+\int_{\Omega}G(u-v)(u-v)=0 \tag{27}$$

Due to hypotheses 1 and 2, $\int_{\Omega}\lambda|\nabla(u-v)|^2+\int_{\Omega}G(u-v)(u-v)\geq 0$. Thus this above equation yields $\frac{d}{dt}\|u-v\|\leq 0$. Therefore the function $t\longmapsto\|u-v\|_{L^2(\Omega)}$ is decreasing. It then follows

$$\|u(t)-v(t)\|_{L^2(\Omega)}\leq\|u(0)-v(0)\|_{L^2(\Omega)}.$$

By equation (6), we can see that $\|u(0)-v(0)\|_{L^2(\Omega)}=0$. Then we deduce for all $t\geq 0$, $\|u(t)-v(t)\|_{L^2(\Omega)}=0$. This establishes the uniqueness of the solution.

4 Concluding Remarks

In this paper, we have stated and demonstrated the existence and uniqueness of the weak solution of the problem (1) - (4). These results will allow us to solve this problem numerically.

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