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Some Identities on the Generalized Changhee-Genocchi Polynomials and Numbers

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Abstract In this paper, we generalize the generating function of the Changhee-Genocchi polynomials. In particular, by means of the method of generating functions and Riordan arrays, we study some properties of the generalized Changhee-Genocchi polynomials. At the same time, we establish some identities between the generalized Changhee-Genocchi polynomials and other combinatorial sequences.

Keywords: Generalized Changhee-Genocchi polynomials, generalized Changhee-Genocchi numbers, generating functions, Riordan arrays.

1 Introduction

In 2016, Byung-Moon Kim first introduced the concept of Changhee-Genocchi polynomials, the Changhee-Genocchi polynomials are defined by the generating function (see[1])

$$\sum_{n=0}^{\infty} CG_n(x)\frac{t^n}{n!} = \frac{2log(1+t)}{2+t}(1+t)^x. \tag{1}$$

When $x = 0$, $CG_n = CG_n(0)$ are called the Changhee-Genocchi numbers.

In addition, Byung-Moon Kim also gived the Changhee-Genocchi polynomials of the order r by the generating function (see[1])

$$\sum_{n=0}^{\infty} CG_n^{(r)}(x)\frac{t^n}{n!} = (\frac{2log(1+t)}{2+t})^r(1+t)^x. \tag{2}$$

For convenience, let us recall some definitions and notations. Here, the generalized Harmonic numbers are defined by the generating function (see[2])

$$\sum_{n=0}^{\infty} H_{n,k,r}(\alpha,\beta)t^n = \frac{(-log(1-\alpha t))^r}{(1-\beta t)^k}. \tag{3}$$

As is well known, the higher-order Changhee numbers are defined by the generating function (see[3])

$$\sum_{n=0}^{\infty} Ch_n^{(k)}\frac{t^n}{n!} = (\frac{2}{2+t})^k. \tag{4}$$

We consider the n-th twisted Daehee polynomials of order k, which are defined by the generating function (see[4])

$$\sum_{n=0}^{\infty} D_{n,\xi}^{(k)}(x)\frac{t^n}{n!} = (\frac{log(1+\xi t)}{\xi t})^k(1+\xi t)^x. \tag{5}$$

In special case, when $x = 0$, $D_{n,\xi}^{(k)}(0) = D_{n,\xi}^{(k)}$ are called twisted Daehee numbers of order r. When $\xi = 1$, $D_{n,1}^{(k)} = D_n^{(k)}$ are higher-order Daehee numbers.

Next, we give several kinds of generating functions which we need in this paper(see[5,6,7,8,9])

$$\sum_{n=0}^{\infty} G_n^{(r)} \frac{t^n}{n!} = (\frac{2t}{e^t+1})^r. \tag{6}$$

$$\sum_{n=0}^{\infty} B_n^{(r)} \frac{t^n}{n!} = (\frac{t}{e^t-1})^r. \tag{7}$$

$$\sum_{n=0}^{\infty} b_n^{(r)} \frac{t^n}{n!} = (\frac{t}{log(1+t)})^r. \tag{8}$$

$$\sum_{n=0}^{\infty} \frac{G_n^{(x)}}{2^n} \frac{t^n}{n!} = (\frac{2}{e^t+1})^x. \tag{9}$$

Let $\Im = \Re[[t]]$ be the ring of the formal power series with real coefficients in some indeterminate t, if $f(t) \in \Im$ and $f(t) = \sum_{n=0}^{\infty} f_n t^n$, let $[t^n]f(t)$ be the coefficient of $[t^n]$ in the formal power series of $f(t)$. If $f(t)$ and $g(t)$ are formal power series, then we get the following relations:

$$[t^n](\alpha f(t) + \beta g(t)) = \alpha[t^n]f(t) + \beta[t^n]g(t). \tag{10}$$

$$\sum_{j=0}^{n} [t^j]f(t)[t^{n-j}]g(t) = [t^n]f(t)g(t). \tag{11}$$

A Riordan array is a couple $D = (d(t), h(t))$ in which $d(t), h(t) \in \Im$ and $h_0 = h(0) = 0$. It defines an infinite lower triangular array $(d_{n,k})_{n,k\in N}$ according to the rule $d_{n,k} = [t^n]d(t)h(t)^k$. So we set $\{d_{n,k}\} = (d(t), h(t))$. Let $D = (d(t), h(t))$ be a Riordan array and $f(t)$ be the generating function of the sequence $\{f_i\}_{i\in N}$, we have (see[10])

$$\sum_{k=0}^{\infty} d_{n,k} f_k = [t^n]d(t)f(h(t)) = [t^n]d(t)[f(y)|_{y=h(t)}]. \tag{12}$$

Recently, many papers have been devoted to the study of the Changhee-Genocchi polynomials and numbers by various methods. In this paper, we generalize the generating function of the Changhee-Genocchi polynomials on the basis of these papers, and investigate some interesting identities related to the generalized Changhee-Genocchi polynomials and numbers.

2 Some Properties of the Generalized Changhee-Genocchi Polynomials

In this paper, we consider the generalized Changhee-Genocchi polynomials which are defined by the generating function

$$\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha, \beta|x) \frac{t^n}{n!} = \frac{2^k log^r(1+\alpha t)}{(2+\beta t)^k}(1+\beta t)^{\alpha x}. \tag{13}$$

where $k \geq 1$, $r \geq 1$ are intergers, α and β are real numbers, and $\alpha\beta \neq 0$.

When $x = 0$, $CG_n^{k,r}(\alpha, \beta|0) = CG_n^{k,r}(\alpha, \beta)$ are called the generalized Changhee-Genocchi numbers.

Particularly, when $\alpha = \beta = 1$ and $k = r$, $CG_n^{r,r}(1, 1|x) = CG_n^{(r)}(x)$, $n \geq 0$.

When $\alpha=\beta=1$ and $k=r=1$, $CG_n^{1,1}(1,1|x)=CG_n(x)$, $n\geq 0$.

In this section, we study some properties of the generalized Changhee-Genocchi polynomials by the generation function method.

For equation (13), we also get

$$\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^n}{n!}=\frac{2^k log^r(1+\alpha t)}{[1+(1+\beta t)]^k}(1+\beta t)^{\alpha x}.$$

Hence, we have

$$\begin{aligned}
2^k log^r(1+\alpha t) &= [1+(1+\beta t)]^k \sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^n}{n!}(1+\beta t)^{-\alpha x}\\
&= \sum_{l=0}^{k}\binom{k}{l}(1+\beta t)^{l-\alpha x}\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^n}{n!}\\
&= \sum_{i=0}^{\infty}\sum_{l=0}^{k}\binom{k}{l}(l-\alpha x)_i\beta^i\frac{t^i}{i!}\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{k}\binom{k}{l}\binom{l-\alpha x}{m}\frac{\beta^m}{(n-m)!}CG_{n-m}^{k,r}(\alpha,\beta|x)t^n. \qquad (14)
\end{aligned}$$

On the other hand, we also can get

$$\begin{aligned}
2^k log^r(1+\alpha t) &= [1+(1+\beta t)]^k \sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^n}{n!}(1+\beta t)^{-\alpha x}\\
&= \sum_{l=0}^{k}\binom{k}{l}(1+\beta t)^{l}\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^n}{n!}\sum_{n=0}^{\infty}\binom{-\alpha x}{n}(-\beta)^n t^n\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=m}^{k}\binom{k}{l}\binom{l}{m}\binom{-\alpha x}{n-m}(-1)^{n-m}\beta^n t^n\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{m=0}^{p}\sum_{l=m}^{k}\binom{k}{l}\binom{l}{m}\binom{-\alpha x}{p-m}CG_{n-p}^{k,r}(\alpha,\beta|x)\frac{(-1)^{p-m}\beta^p}{(n-p)!}t^n. \qquad (15)
\end{aligned}$$

Theorem 1. Let n be nonnegative integers, $k\geq 1$ and $r\geq 1$ are integers, we have

$$\begin{aligned}
&\sum_{p=0}^{n}\sum_{m=0}^{p}\sum_{l=m}^{k}\binom{k}{l}\binom{l}{m}\binom{-\alpha x}{p-m}CG_{n-p}^{k,r}(\alpha,\beta|x)\frac{(-1)^{p-m}\beta^p}{(n-p)!}\\
&= \sum_{m=0}^{n}\sum_{l=0}^{k}\binom{k}{l}\binom{l-\alpha x}{m}CG_{n-m}^{k,r}(\alpha,\beta|x)\frac{\beta^m}{(n-m)!}\\
&= 2^k r!\alpha^n\frac{B_{n,r}(1,-1,2!,-3!,...)}{n!}. \qquad (16)
\end{aligned}$$

Proof By comparing the coefficients of t^n on both sides of the equation (14) and (15), theorem 1 is proved.

Theorem 2. Let n be nonnegative integers, $k,r,p,q\geq 1$ are integers, we have

$$\sum_{j=0}^{n}\sum_{m=0}^{j}\sum_{l=0}^{k}\binom{k}{l}\binom{l-\alpha x}{m}\frac{CG_{j-m}^{k,r}(\alpha,\beta|x)}{(j-m)!}\frac{CG_{n-j}^{p,q}(\alpha,\beta|y)}{(n-j)!}n!\beta^m=2^k CG_n^{p,q+r}(\alpha,\beta|y). \qquad (17)$$

Proof By equation (9) and equation (13), we get

$$\sum_{j=0}^{n}\sum_{m=0}^{j}\sum_{l=0}^{k}\binom{k}{l}\binom{l-\alpha x}{m}\frac{\beta^m}{(j-m)!}CG_{j-m}^{k,r}(\alpha,\beta|x)\frac{CG_{n-j}^{p,q}(\alpha,\beta|y)}{(n-j)!}$$
$$=\sum_{j=0}^{n}[t^j]2^k log^r(1+\alpha t)[t^{n-j}]\frac{2^p log^q(1+\alpha t)}{(2+\beta t)^p}(1+\beta t)^{\alpha y}$$
$$=[t^n]2^k\frac{2^p log^{r+q}(1+\alpha t)}{(2+\beta t)^p}(1+\beta t)^{\alpha y}=\frac{2^k}{n!}CG_n^{p,r+q}(\alpha,\beta|y).$$

Theorem 3. Let n be nonnegative integers, $k,r\geq 1$ are integers, we have

$$\sum_{m=0}^{n}\binom{k}{m}(\frac{\beta}{2})^m CG_{n-m}^{k,r}(\alpha,\beta)\frac{n!}{(n-m)!}=r!\alpha^n S_1(n,r). \tag{18}$$

Proof By equation (13), when $x=0$, we have

$$\sum_{n=0}^{\infty}CG_n^{k,r}(\alpha,\beta)\frac{t^n}{n!}=\frac{2^k log^r(1+\alpha t)}{(2+\beta t)^k}.$$

Hence, we can get

$$(1+\frac{\beta}{2}t)^k\sum_{n=0}^{\infty}CG_n^{k,r}(\alpha,\beta)\frac{t^n}{n!}=log^r(1+\alpha t).$$

Here, we simpify the left side of this equation

$$(1+\frac{\beta}{2}t)^k\sum_{n=0}^{\infty}CG_n^{k,r}(\alpha,\beta)\frac{t^n}{n!}=\sum_{l=0}^{\infty}\binom{k}{l}(\frac{\beta}{2})^l t^l\sum_{n=0}^{\infty}CG_n^{k,r}(\alpha,\beta)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{k}{m}(\frac{\beta}{2})^m CG_{n-m}^{k,r}(\alpha,\beta)\frac{t^n}{(n-m)!}.$$

For the right side, we have (see[11])

$$log^r(1+\alpha t)=\sum_{n=r}^{\infty}r!\alpha^n S_1(n,r)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$, theorem 3 is proved.

Corollary 1. In theorem 3, when $k=1$ and $n\geq 1$, we get

$$CG_n^{1,r}(\alpha,\beta)+\frac{n\beta}{2}CG_{n-1}^{1,r}(\alpha,\beta)=r!\alpha^n S_1(n,r).$$

Corollary 2. In theorem 3, when $\alpha=\beta=1$, and $k=r=1$, we get theoerm 11 of the reference [1].

Theorem 4. Let n be nonegative integers, $k\geq 1$, $r\geq 2$ are integers, we have

$$CG_{n+1}^{k,r}(\alpha,\beta|x)+\frac{k\beta}{2}CG_n^{k+1,r}(\alpha,\beta|x)$$
$$=\sum_{m=0}^{n}\binom{n}{m}[\frac{(n-m)!}{(-\alpha)^{m-n}}\alpha r CG_m^{k,r-1}(\alpha,\beta|x)+x\beta^{n-m+1}(\alpha)_{n-m+1}CG_m^{k,r}(\alpha,\beta|x-1)]. \tag{19}$$

Proof Let's take the derivative about t, on both sides of equation (13), we get

$$\sum_{n=1}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^{n-1}}{(n-1)!}$$
$$= [\frac{-k\beta}{2}\frac{2^{k+1}log^r(1+\alpha t)}{(2+\beta t)^{k+1}} + \frac{2^k log^{r-1}(1+\alpha t)}{(2+\beta t)^k}\frac{\alpha r}{1+\alpha t}](1+\beta t)^{\alpha x} + \frac{2^k log^r(1+\alpha t)}{(2+\beta t)^k}\alpha\beta x(1+\beta t)^{\alpha x-1}$$
$$= \frac{-k\beta}{2}\sum_{n=0}^{\infty} CG_n^{k+1,r}(\alpha,\beta|x)\frac{t^n}{n!} + \alpha r\sum_{n=0}^{\infty} CG_n^{k,r-1}(\alpha,\beta|x)\frac{t^n}{n!}\sum_{n=0}^{\infty}(-\alpha)^n t^n$$
$$+ \alpha\beta x\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x-1)\frac{t^n}{n!}\sum_{n=0}^{\infty}(\alpha-1)_n\beta^n\frac{t^n}{n!}$$
$$= \frac{-k\beta}{2}\sum_{n=0}^{\infty} CG_n^{k+1,r}(\alpha,\beta|x)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}[(n-m)!(-1)^{n-m}\alpha^{n-m+1}rCG_m^{k,r-1}(\alpha,\beta|x)$$
$$+ x\beta^{n-m+1}(\alpha)_{n-m+1}CG_m^{k,r}(\alpha,\beta|x-1)]\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of this equation, theorem 4 is proved.

Theorem 5. Let n be nonegative integers, $k, r \geq 1$ and $m, l \geq 1$ are integers, we have

$$\sum_{p=0}^{n}\binom{n}{p}CG_p^{k,r}(\alpha,\beta|x)Ch_{n-p}^{(m)}(\alpha y)\beta^{n-p} = CG_n^{k+m,r}(\alpha,\beta|x+y), \quad (20)$$

$$\sum_{p=0}^{n}\binom{n}{p}CG_p^{k,r}(\alpha,\beta|x)CG_{n-p}^{m,l}(\alpha,\beta|y) = CG_n^{k+m,r+l}(\alpha,\beta|x+y). \quad (21)$$

Proof

$$\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^n}{n!}\sum_{n=0}^{\infty} Ch_n^{(m)}(\alpha y)\frac{(\beta t)^n}{n!} = \sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}CG_p^{k,r}(\alpha,\beta)Ch_{n-p}^{(m)}(\alpha y)\beta^{n-p}\frac{t^n}{n!}$$
$$= \frac{2^k log^r(1+\alpha t)}{(2+\beta t)^k}(\frac{2}{2+\beta t})^m(1+\beta t)^{\alpha(x+y)} = \sum_{n=0}^{\infty} CG_n^{k+m,r}(\alpha,\beta|x+y)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of this equation, equation (20) is proved. The proof of (21) is similar to that of (20).

3 Identities Involving the Generalized Changhee-Genocchi Polynomials and Numbers

In this section, we establish some identities which are related to the generalized Changhee-Genocchi polynomials. Then we find the Riordan array of the generalized Changhee-Genocchi numbers, and give several identities by means of the Riordan arrays.

Theorem 6. Let n be nonegative integers, $k, r, p \geq 1$ are integers, we have

$$\sum_{m=0}^{n} Ch_m^{(k)}(\alpha x+p)\beta^m n!(-1)^{n-m+r}H_{n-m,p,r}(\alpha,\beta) = CG_n^{k,r}(\alpha,\beta|x). \quad (22)$$

Proof

$$\begin{aligned}\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^n}{n!} &= (\frac{2}{2+\beta t})^k(1+\beta t)^{(\alpha x+p)}\frac{log^r(1+\alpha t)}{(1+\beta t)^p}\\ &= \sum_{n=0}^{\infty} Ch_n^{(k)}(\alpha x+p)\beta^n\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^{n+r}H_{n,p,r}(\alpha,\beta)t^n\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} Ch_m^{(k)}(\alpha x+p)\beta^m(-1)^{n-m+r}H_{n-m,p,r}(\alpha,\beta)\frac{t^n}{m!}.\end{aligned}$$

By comparing the coefficients of t^n on both sides of this equation, theorem 6 is proved.

Corollary 3. In theorem 6, when $\alpha=\beta$, and $p=1$, the following relations hold:

$$\sum_{m=0}^{n}\binom{n}{m}\frac{\alpha^n(-1)^{n-m+r}}{(n-m)!}Ch_m^{(k)}(\alpha x+1)H(n-m,r-1) = CG_n^{k,r}(\alpha,\alpha|x). \tag{23}$$

Theorem 7. Let $n\geq r$, $k\geq 1$ and $r\geq 1$ are integers, then

$$\sum_{m=0}^{n-r}\binom{n-r}{m}Ch_m^{(k)}(\alpha x)\beta^m\alpha^r D_{n-r-m,\alpha}^{(r)}\frac{n!}{(n-r)!} = CG_n^{k,r}(\alpha,\beta|x). \tag{24}$$

Proof According to the equation (4) and (5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta|x)\frac{t^n}{n!} &= \frac{2^k log^r(1+\alpha t)}{(2+\beta t)^k}(1+\beta t)^{\alpha x}\\ &= (\frac{2}{2+\beta t})^k(1+\beta t)^{\alpha x}\frac{log^r(1+\alpha t)}{(\alpha t)^r}(\alpha t)^r\\ &= \sum_{n=0}^{\infty} Ch_n^{(k)}(\alpha x)\frac{(\beta t)^n}{n!}\sum_{n=0}^{\infty} D_{n,\alpha}^{(r)}\frac{t^n}{n!}(\alpha t)^r\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}Ch_m^{(k)}(\alpha x)\beta^m\alpha^r D_{n-m,\alpha}^{(r)}\frac{\alpha^r t^{n+r}}{n!}.\end{aligned}$$

By comparing the coefficients of t^n on both sides of this equation, theorem 7 is proved.

Corollary 4. In theorem 7, when $\alpha=1$, the following relation holds:

$$\sum_{m=0}^{n-r}\binom{n-r}{m}Ch_m^{(k)}(x)\beta^m D_{n-r-m}^{(r)}\frac{n!}{(n-r)!} = CG_n^{k,r}(1,\beta|x). \tag{25}$$

Corollary 5. In theorem 7, when $\alpha=\beta=1$, and $k=r$, the following relation holds:

$$\sum_{m=0}^{n-r}\binom{n-r}{m}Ch_m^{(r)}(x)D_{n-r-m}^{(r)}\frac{n!}{(n-r)!} = CG_n^{(r)}(x). \tag{26}$$

Theorem 8. Let $n\geq\min\{r,s\}$, $k,r,s\geq 1$ are integers, we have

$$\sum_{m=0}^{n}\alpha^{n-m}b_{n-m}^{(s)}\frac{CG_m^{k,r}(\alpha,\beta|x)}{m!(n-m)!} = \begin{cases} CG_{n-s}^{k,r-s}(\alpha,\beta|x)\frac{\alpha^s}{(n-s)!}, & r>s\\ Ch_{n-s}^{(k)}(\alpha x)\alpha^s\beta^{n-s}\frac{1}{(n-s)!}, & r=s\\ \sum_{m=0}^{n-r}\binom{n-r}{m}\frac{\beta^m\alpha^{n-m}}{(n-r)!}Ch_m^{(k)}(\alpha x)b_{n-r-m}^{(s-r)}. & r<s\end{cases} \tag{27}$$

Proof By equation (8),(11),(13), we have

$$\sum_{m=0}^{n} b_{n-m}^{(s)} \frac{CG_m^{k,r}(\alpha,\beta|x)}{\binom{n}{m}^{-1}\alpha^{m-n}n!} = [t^n]\alpha^s t^s \frac{2^k log^{r-s}(1+\alpha t)}{(2+\beta t)^k}$$

$$= \begin{cases} CG_{n-s}^{k,r-s}(\alpha,\beta)\frac{\alpha^s}{(n-s)!}, & r>s \\ Ch_{n-s}^{(k)}(\alpha x)\alpha^s\beta^{n-s}\frac{1}{(n-s)!}, & r=s \\ \sum_{m=0}^{n-r}\binom{n-r}{m}\frac{\beta^m\alpha^{n-m}}{(n-r)!}Ch_m^{(k)}(\alpha x)b_{n-r-m}^{(s-r)}. & r<s \end{cases}$$

Theorem 9. Let $k, r \geq 1$ and $p, l \geq 0$ be integers, we have

$$\sum_{j=0}^{r}\sum_{p=0}^{j}\sum_{l=0}^{r-j}\binom{r}{j}\binom{k}{l}\beta^l \frac{s_2(j,p)s_2(r-j,l)}{2^l\alpha^{p+l}} CG_p^{k,r}(\alpha,\beta)\frac{l!}{r!} = \delta_{n,r}, \tag{28}$$

where $\delta_{n,r}$ is the Kronecker delta symbol.

Proof Replacing t by $\frac{e^t-1}{\alpha}$ in equation (13), we have

$$\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta)\alpha^{-n}\frac{(e^t-1)^n}{n!} = \frac{2^k t^r}{[2+\frac{\beta(e^t-1)}{\alpha}]^k}.$$

Hence, we have

$$\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta)\alpha^{-n}\frac{(e^t-1)^n}{n!}[2+\frac{\beta(e^t-1)}{\alpha}]^k = 2^k t^r. \tag{29}$$

As is well known, $\sum_{n=k}^{\infty} s_2(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}$ (see[12]).

Now, we consider the left-hand side of the equation (29), we have

$$\sum_{n=0}^{\infty} CG_n^{k,r}(\alpha,\beta)\alpha^{-n}\frac{(e^t-1)^n}{n!}[2+\frac{\beta(e^t-1)}{\alpha}]^k$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m} CG_n^{k,r}(\alpha,\beta)\alpha^{-n}s_2(m,n)\frac{t^m}{m!}\sum_{i=0}^{\infty}\sum_{l=0}^{i}\binom{k}{l}2^{k-l}(\frac{\beta}{\alpha})^l l! s_2(i,l)\frac{t^i}{i!}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{p=0}^{j}\sum_{l=0}^{n-j}\binom{n}{j}\binom{k}{l}\beta^l\frac{s_2(j,p)s_2(n-j,l)}{2^{l-k}\alpha^{p+l}}CG_p^{k,r}(\alpha,\beta)\frac{l!}{n!}t^n.$$

By comparing the coefficients of t^r, theorem 9 is proved.

By the concept of Riordan arrays and equation (13), we get $\{\frac{CG_n^{k,r}(\alpha,\beta)}{n!}\} = (2^k(2+\beta t)^{-k}, ln(1+\alpha t))$, we can get the follwing results:

Theorem 10. Let $n, k \geq 1$ be integers, we have

$$\sum_{j=1}^{n} CG_n^{k,j}(\alpha,\beta)\frac{1}{j!} = n\alpha\beta^{n-1}Ch_{n-1}^{(k)}. \tag{30}$$

Proof

$$\sum_{j=1}^{n} CG_n^{k,j}(\alpha,\beta)\frac{1}{n!}\frac{1}{j!} = [t^n]2^k(2+\beta t)^{-k}[e^y-1|_{y=log(1+\alpha t)}]$$

$$= [t^{n-1}]2^k\alpha(2+\beta t)^{-k} = [t^{n-1}]\alpha\sum_{n=0}^{\infty}Ch_n^{(k)}\beta^n\frac{t^n}{n!}$$

$$= \alpha\beta^{n-1}\frac{Ch_{n-1}^{(k)}}{(n-1)!}.$$

Hence, the identity (30) can be obtained immediately.

Theorem 11. Let $n, k, j, m \geq 1$ be integers, we set up the following equation:

$$\sum_{j=1}^{n} CG_n^{k,j}(\alpha,\beta)\frac{G_j^{(m)}}{n!j!} = \sum_{l=0}^{n}\binom{n-l+m-1}{m-1}\frac{(-\alpha)^{n-l}}{2^{n-l}l!}CG_l^{k,m}(\alpha,\beta), \tag{31}$$

$$\sum_{j=1}^{n} CG_n^{k,j}(\alpha,\beta)\frac{G_j^{(x)}}{2^j n!j!} = \sum_{l=0}^{n}\binom{l+k-1}{k-1}\binom{-x}{n-l}(-\frac{1}{2})^n\alpha^{n-l}\beta^l. \tag{32}$$

Proof

$$\sum_{j=1}^{n} CG_n^{k,j}(\alpha,\beta)\frac{1}{n!}\frac{G_j^{(m)}}{j!} = [t^n]2^k(2+\alpha t)^{-k}[(\frac{2y}{e^y+1})^m|_{y=log(1+\alpha t)}]$$

$$= [t^n]\frac{2^k log^m(1+\alpha t)}{(2+\beta t)^k}\frac{2^m}{(2+\alpha t)^m} = [t^n]\sum_{n=0}^{\infty} CG_n^{k,m}(\alpha,\beta)\frac{t^n}{n!}\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}(-\frac{\alpha}{2})^n t^n$$

$$= \sum_{l=0}^{n}\binom{n-l+m-1}{m-1}\frac{(-\alpha)^{n-l}}{2^{n-l}l!}CG_l^{k,m}(\alpha,\beta).$$

Hence, the equation (31) is proved. The proof of (32) is similar to that of (31), and it is omitted here.

Corollary 6. In theorem 11, when $\alpha = \beta$, the following relations hold:

$$\sum_{j=1}^{n} CG_n^{k,j}(\alpha,\alpha)\frac{G_j^{(m)}}{j!} = CG_n^{k+m,m}(\alpha,\alpha),$$

$$\sum_{j=1}^{n} CG_n^{k,j}(\alpha,\beta)\frac{G_j^{(x)}}{2^j n!j!} = (-\frac{\alpha}{2})^n\binom{n+k+x-1}{n}.$$

Theorem 12. Let $n, k, j, m \geq 1$ be integers, we set the following equations:

$$\sum_{j=1}^{n} CG_n^{k,j}(\alpha,\beta)\binom{n+m}{n}\frac{m!B_j^{(m)}}{j!} = \alpha^{-m}CG_{n+m}^{k,m}(\alpha,\beta). \tag{33}$$

Proof

$$\sum_{j=1}^{n} CG_n^{k,j}(\alpha,\beta)\frac{1}{n!}\frac{B_j^{(m)}}{j!} = [t^n]2^k(2+\alpha t)^{-k}[(\frac{y}{e^y-1})^m|_{y=log(1+\alpha t)}]$$

$$= [t^n]\frac{2^k log^m(1+\alpha t)}{(2+\beta t)^k}(\alpha t)^{-m} = CG_{n+m}^{k,m}(\alpha,\beta)\frac{\alpha^{-m}}{(n+m)!}.$$

Hence, theoerm 12 can be obtained immediately.

The second and third sections are our main results. We generalize the generating function of the Changhee-Genocchi polynomials and find some new identities by the method of generating functions and Riordan arrays. Specially, these identities contain some relations about classical Changhee-Genocchi polynomials. In addition, it is easy to see that combinations of special sequences can be represented by the generalized Changhee-Genocchi polynomials, such as the Changhee polynomials and the generalized Harmonic numbers, the Changhee polynomials and the Daehee numbers, etc.

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