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A Geometric Integration Based on Magnus Series Expansion for Human T-Cell Lymphotropic Virus I (HTLV-I) Infection of CD4$^+$ T-Cells Model

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Abstract In this paper, we investigated a numeric integration based on Magnus series expansion namely Magnus Series Expansion Method (NMG) for nonlinear Human T-Cell Lymphotropic Virus I (HTLV-I) infection of CD4$^+$ T-cells model. Fourth order Magnus series expansion method (NMG4) and explicit Runge-Kutta (RK45) are used to obtain numerical solutions of HTLV-I infection of CD4$^+$ T-cells model. The results obtained by NMG4 and RK45 are compared.

Keywords: Non-linear Differential Equation System, Magnus Series Expansion Method, Geometric Integration, Lie Group Method.

1 Introduction

Dynamics of Human T-Cell Lymphotropic Virus I (HTLV-I) infection of CD4$^+$ T-cells model examined in [1-7] is,

$$\frac{dT}{dt} = \lambda - \mu_T T - kT_A T \tag{1}$$
$$\frac{dT_L}{dt} = kT_A T - (\mu_L + \alpha)T_L \tag{2}$$
$$\frac{dT_A}{dt} = \alpha T_L - (\mu_A + \rho)T_A \tag{3}$$
$$\frac{dT_M}{dt} = \rho T_A + \beta T_M(1 - T_M/T_{max}) - \mu_M T_M \tag{4}$$

with the initial conditions,

$$T(0) = P_1, \quad T_L(0) = P_2, \quad T_A(0) = P_3, \quad T_M(0) = P_4, \tag{5}$$

where $T(t), T_L(t), T_A(t), T_M(t)$ denote the concentration of healthy CD4$^+$ T-cells at time t, the concentration of latently infected CD4$^+$ T-cells, the concentration of actively infected CD4$^+$ T-cells and the concentration of leukemic cells at time t respectively. The parameters λ, μ_T and k denote the natural date rate of CD4$^+$ T-cells, the rate at which uninfected cells are contacted by actively infected cells, the rate of infection of T-cells with virus from actively infected cells respectively. μ_L, μ_A, μ_M are blanket death terms for latently infected, actively infected and leukemic cells respectively. α and ρ denote the rates at which latently infected and actively infected cells become actively infected and leukemic respectively. The rate β determines the the speed at which the saturation level for leukemia cells is reached. T_{max} is the maximal value that adult T-cell leukemia can reach [7].

The purpose of this paper is to obtain numerical solution of the system (1-4) subject to the initial conditions (5) by using a structure preserving numerical integrator based on Magnus Expansion, namely Magnus Expansion Method.

In 1954, Magnus [8] provided an exponential representation of the solution of a first order linear homogeneous differential equation for a linear operator that was named Magnus Expansion after him.

Since 1960's the Magnus expansion has been successfully applied in various areas of physics and chemistry (see [9] for a list of references). Iserles and Norsett (1997) [10,11] presented a practical recursive algorithm that generated the terms of Magnus expansion. Blanes et al. (1998) [12] considered the approximate solutions of matrix linear differential equations by matrix exponentials and the convergence issue of Magnus and Fer expansions. They obtained the upper bounds for the convergence radius in terms of the norm of the defining matrix of the system. Moan and Niesen (2008) [13] considered the question: When does the series converge? The main result they obtained, established a necessary condition for convergence.

2 Magnus Expansion Method

The linear differential equation on a matrix Lie-group is an equation of the form

$$Y' = A(t)Y, \qquad t \geq 0, \qquad Y(0) = Y_0 \in \mathcal{G}, \tag{6}$$

where $A : \mathcal{G} \longrightarrow \mathcal{R}$ is the matrix function, $\mathcal{G}$ is the Lie group, g is the Lie algebra of the corresponding Lie-group $\mathcal{G}$. Magnus expressed the solution of equation (6) as the exponential of a certain function [8],

$$Y(t) = e^{\Omega(t)}, \tag{7}$$

and obtained an infinite recursive series for Ω as follows,

$$\Omega_0 \equiv 0, \tag{8}$$

$$\Omega_{n+1} = \int_0^t dexp_{\Omega_n}^{-1} A(\xi) d\xi = \sum_{k=0}^{\infty} \frac{B_k}{k!} \int_0^t ad_{\Omega_n}^k A(\xi) d\xi, \quad n = 0, 1, 3, \ldots \tag{9}$$

where B_k are the Bernoulli numbers. Substituting the equation (8) into the equation (9) one can get the Ω_i for $i = 0, 1, 2, 3, \ldots$, respectively

$$\begin{aligned}
\Omega_1 &= \int_0^t A(t_1) dt_1 \\
\Omega_2 &= \int_0^t A(t_1) dt_1 - \frac{1}{2} \int_0^t [\int_0^{t_2} A(t_1) dt_1, A(t_2)] dt_2 + \ldots \\
\Omega_3 &= \int_0^t A(t_1) dt_1 - \frac{1}{2} \int_0^t [\int_0^{t_2} A(t_1) dt_1, A(t_2)] dt_2 \\
&\quad + \frac{1}{12} \int_0^t [\int_0^{t_3} A(t_1) dt_1, [\int_0^{t_3} A(t_1) dt_1, A(t_3)]] dt_3 \\
&\quad + \frac{1}{4} \int_0^t [\int_0^{t_3} [\int_0^{t_2} A(t_1) dt_1, A(t_2)] dt_2, A(t_3)] dt_3 + \ldots
\end{aligned} \tag{10}$$

The Magnus series expansion is,

$$\Omega(t) = \sum_{k=0}^{\infty} H_k(t), \tag{11}$$

where each H_k includes exactly $k+1$ integrals and k commutators [14].

Thus,

$$\begin{aligned}
H_1 &= \int_0^t A(t_1) dt_1 \\
H_2 &= -\frac{1}{2} \int_0^t [\int_0^{t_2} A(t_1) dt_1, A(t_2)] dt_2
\end{aligned}$$

$$
\begin{aligned}
H_3 = &\frac{1}{12}\int_0^t [\int_0^{t_3} A(t_1)dt_1, [\int_0^{t_3} A(t_1)dt_1, A(t_3)]]dt_3 \\
&+\frac{1}{4}\int_0^t [\int_0^{t_3} [\int_0^{t_2} A(t_1)dt_1, A(t_2)]dt_2, A(t_3)]dt_3 \\
H_4 = &-\frac{1}{24}\int_0^t [\int_0^{t_4} A(t_1)dt_1, [\int_0^{t_4} [\int_0^{t_2} A(t_1)dt_1, A(t_2)]dt_2, A(t_4)]]dt_4 \\
&-\frac{1}{24}\int_0^t [\int_0^{t_4} [\int_0^{t_2} A(t_1)dt_1, A(t_2)]dt_2, [\int_0^{t_4} A(t_1)dt_1, A(t_4)]]dt_4 \\
&-\frac{1}{24}\int_0^t [\int_0^{t_4} [\int_0^{t_3} A(t_1)dt_1, [\int_0^{t_3} A(t_1)dt_1, A(t_3)]]dt_3, A(t_4)]dt_4 \\
&-\frac{1}{8}\int_0^t [\int_0^{t_4} [\int_0^{t_3} [\int_0^{t_2} A(t_1)dt_1, A(t_2)]dt_2, A(t_3)]dt_3, A(t_4)]dt_4.
\end{aligned}
$$

By using multivariate Gaussian quadrature, Casas & Iserles [15] presented algorithms for third and fourth order Magnus Expansion Method (NMG3 and NMG4) for nonlinear Lie type differential equations as follows, Third order Magnus Expansion Method for nonlinear equations (NMG3)

$$
\begin{aligned}
u_1 &= 0 \\
k_1 &= hA(0, Y_0) \\
u_2 &= \frac{1}{2}k_1 \\
k_2 &= hA(\frac{h}{2}, e^{u_2}Y_0) \\
u_3 &= \frac{1}{4}(k_1 + k_2) \\
k_3 &= hA(\frac{h}{2}, e^{u_3}Y_0) \\
u_4 &= k_2 \\
k_4 &= hA(h, e^{u_4}Y_0) \\
v_3 &= \frac{1}{6}(k_1 + 4k_3 + k_4) - \frac{1}{3}[u_3, k_3] - \frac{1}{12}[u_4, k_4] \\
Y_1(t) &= e^{v_3}Y_0.
\end{aligned} \tag{12}
$$

Fourth order Magnus Expansion Method for nonlinear equations (NMG4)

$$
\begin{aligned}
u_1 &= 0 \\
k_1 &= hA(t_n, Y_n) \\
Q_1 &= k_1 \\
u_2 &= \frac{1}{2}Q_1 \\
k_2 &= hA(t_n + \frac{h}{2}, e^{u_2}Y_0) \\
Q_2 &= k_2 - k_1 \\
u_3 &= \frac{1}{2}Q_1 + \frac{1}{4}Q_2 \\
k_3 &= hA(t_n + \frac{h}{2}, e^{u_3}Y_0) \\
Q_3 &= k_3 - k_2 \\
u_4 &= Q_1 + Q_2
\end{aligned}
$$

$$
\begin{aligned}
k_4 &= hA(t_n + h, e^{u_4} Y_0)\\
Q_4 &= k_4 - 2k_2 + k_1\\
u_5 &= \frac{1}{2}Q_1 + \frac{1}{4}Q_2 + \frac{1}{3}Q_3 - \frac{1}{24}Q_4 - \frac{1}{48}[Q_1, Q_2]\\
k_5 &= hA(t_n + \frac{h}{2}, e^{u_5} Y_0)\\
Q_5 &= k_5 - k_2\\
u_6 &= Q_1 + Q_2 + \frac{2}{3}Q_5 + \frac{1}{6}Q_4 - \frac{1}{6}[Q_1, Q_2]\\
k_6 &= hA(t_n + h, e^{u_6} Y_0)\\
Q_6 &= k_6 - 2k_2 + k_1\\
v &= Q_1 + Q_2 + \frac{2}{3}Q_5 + \frac{1}{6}Q_6 - \frac{1}{6}[Q_1, Q_2 - Q_3 + Q_5 + \frac{1}{2}Q_6]\\
Y_{n+1} &= e^{v} Y_n.
\end{aligned}
\tag{13}
$$

3 Numerical Results

In this section, NMG4 and the fourth order explicit RungeâĂŞKutta (RK45) are applied to the Human T-Cell Lymphotropic Virus I (HTLV-I) infection of CD4$^+$ T-cells model by Mathematica 7. Throughout this paper we set $\mu_T = 0.66mm^3/day, \mu_L = 0.06day, \mu_A = 0.05day, \mu_M = 0.005day$, $k = 0.5$, $\lambda = 0.6$, $\alpha = 0.004day$, $\beta = 0.0003day$, $\rho = 0.00004day$ and $T_{max} = 2200mm^3$.

$$\frac{dT}{dt} = \lambda - \mu_T T - kT_A T \tag{14}$$

$$\frac{dT_L}{dt} = kT_A T - (\mu_L + \alpha)T_L \tag{15}$$

$$\frac{dT_A}{dt} = \alpha T_L - (\mu_A + \rho)T_A \tag{16}$$

$$\frac{dT_M}{dt} = \rho T_A + \beta T_M(1 - T_M/T_{max}) - \mu_M T_M \tag{17}$$

with the initial conditions,

$$T(0) = 1000, \quad T_L(0) = 250, \quad T_A(0) = 1.5, \quad T_M(0) = 0, \tag{18}$$

By using the following transformation [16],

$$T = p, \quad T_L = q, \quad T_A = r, \quad T_M = s, \tag{19}$$

the equation system (14-17) yields the Lie-type matrix equation,

$$\begin{bmatrix} p\\ q\\ r\\ s\\ 1 \end{bmatrix}' = \begin{bmatrix} -\mu_T - kr & 0 & 0 & 0 & \lambda\\ kr & -\mu_L - \alpha & 0 & 0 & 0\\ 0 & \alpha & -\mu_A - \rho & 0 & 0\\ 0 & 0 & \rho & \beta(1 - \frac{s}{T_{max}}) - \mu_A & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p\\ q\\ r\\ s\\ 1 \end{bmatrix}, \tag{20}$$

where prime denotes derivative with respect to time t.

If we call $\begin{bmatrix} p\\ q\\ r\\ s\\ 1 \end{bmatrix}$ as Y and $\begin{bmatrix} -\mu_T - kr & 0 & 0 & 0 & \lambda\\ kr & -\mu_L - \alpha & 0 & 0 & 0\\ 0 & \alpha & -\mu_A - \rho & 0 & 0\\ 0 & 0 & \rho & \beta(1 - \frac{s}{T_{max}}) - \mu_A & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ as A, then equation system (14-17) takes the form,

$$Y' = AY. \tag{21}$$

By using the algorithm (13) and conducting the iteration for the time interval (0,1000) with the step size $\frac{1}{10}$, the following results are obtained. Table 1-4. compare the NMG4 and RK45 solutions for the time interval (0,20) with the step size $\frac{1}{10}$ for the equation system (14-17). The obtained results for $T(t), T_L(t), T_A(t)$ and $T_M(t)$ are given in Fig. 1-13.

Table 1. NMG4 and RK45 solutions of $T(t)$ for the time interval (0,20) with the step size $\frac{1}{10}$ for the equation system (14-17).

t	NMG4 solution	RK45 solution	Absolute Difference
0	1000	1000	0
2	7.13582	7.13582	4.12047×10^{-6}
4	0.096573	0.096573	6.62869×10^{-6}
6	0.0723808	0.0723808	6.33054×10^{-7}
8	0.0610988	0.0610988	5.42661×10^{-7}
10	0.0549001	0.0549001	4.25295×10^{-7}
12	0.0513063	0.0513063	3.03046×10^{-7}
14	0.049261	0.049261	1.8866×10^{-7}
16	0.048239	0.048239	8.85135×10^{-8}
18	0.047952	0.047952	4.96236×10^{-9}
20	0.0482307	0.0482307	5.93419×10^{-8}

Table 2. NMG4 and RK45 solutions of $T_L(t)$ for the time interval (0,20) with the step size $\frac{1}{10}$ for the equation system (14-17).

t	NMG4 solution	RK45 solution	Absolute Difference
0	250	250	0
2	796.75	796.75	7.44233×10^{-5}
4	707.304	707.304	6.10276×10^{-5}
6	623.37	623.37	5.38087×10^{-5}
8	549.53	549.53	4.73637×10^{-5}
10	484.566	484.566	4.16416×10^{-5}
12	427.411	427.411	3.6581×10^{-5}
14	377.125	377.125	3.21183×10^{-5}
16	332.882	332.882	2.81897×10^{-5}
18	293.955	293.955	2.47478×10^{-5}
20	259.704	259.703	2.17017×10^{-5}

Table 3. NMG4 and RK45 solutions of $T_A(t)$ for the time interval (0,20) with the step size $\frac{1}{10}$ for the equation system (14-17).

t	NMG4 solution	RK45 solution	Absolute Difference
0	1.5	1.5	
2	6.39744	6.39744	1.79348×10^{-7}
4	11.5167	11.5167	6.37668×10^{-7}
6	15.4728	15.4728	1.00891×10^{-7}
8	18.453	18.453	1.29307×10^{-6}
10	20.6224	20.6224	1.50456×10^{-6}
12	22.1215	22.1215	1.65579×10^{-6}
14	23.0698	23.0698	1.75723×10^{-6}
16	23.5689	23.5689	1.81803×10^{-6}
18	23.7046	23.7046	1.84408×10^{-6}
20	23.5496	23.5496	1.84767×10^{-6}

Table 4. NMG4 and RK45 solutions of $T_M(t)$ for the time interval (0,20) with the step size $\frac{1}{10}$ for the equation system (14-17).

t	NMG4 solution	RK45 solution	Absolute Difference
0	0	0	0
2	0.000290763	0.000290763	5.77703×10^{-11}
4	0.00100986	0.00100986	2.48410×10^{-11}
6	0.00208226	0.00208226	4.16242×10^{-11}
8	0.00341953	0.00341953	1.33443×10^{-10}
10	0.00494826	0.00494826	2.44045×10^{-10}
12	0.00660783	0.00660783	3.67955×10^{-10}
14	0.00834854	0.00834854	5.00714×10^{-10}
16	0.01013	0.01013	6.38569×10^{-10}
18	0.0119194	0.0119194	7.79346×10^{-10}
20	0.0136909	0.0136909	9.17819×10^{-10}

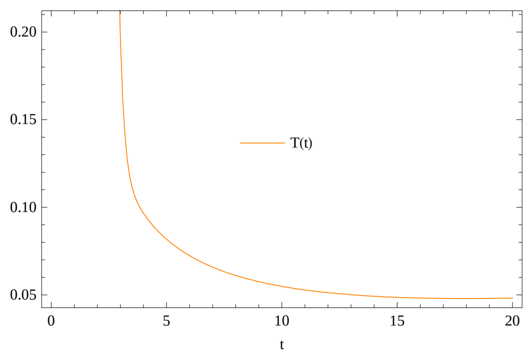

Figure 1. NMG4 solutions of $T(t)$ for the time interval (0,20) with the step size $\frac{1}{10}$ for the equation system (14-17)

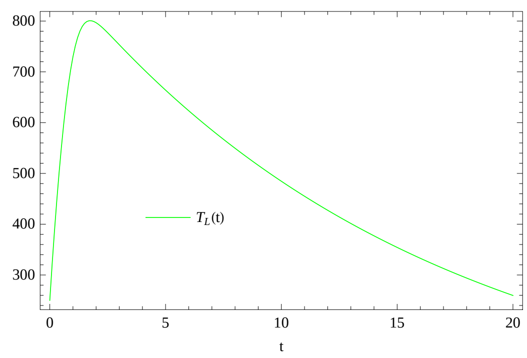

Figure 2. NMG4 solutions of $T_L(t)$ for the time interval (0,20) with the step size $\frac{1}{10}$ for the equation system (14-17)

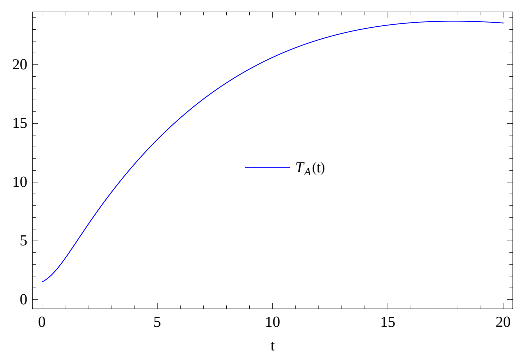

Figure 3. NMG4 solutions of $T_A(t)$ for the time interval (0,20) with the step size $\frac{1}{10}$ for the equation system (14-17)

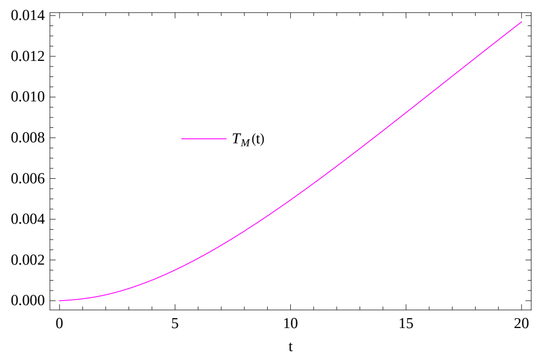

Figure 4. NMG4 solutions of $T_M(t)$ for the time interval (0,20) with the step size $\frac{1}{10}$ for the equation system (14-17)

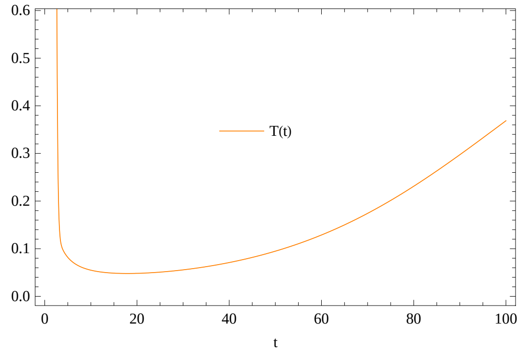

Figure 5. NMG4 solutions of $T(t)$ for the time interval (0,100) with the step size $\frac{1}{10}$ for the equation system (14-17)

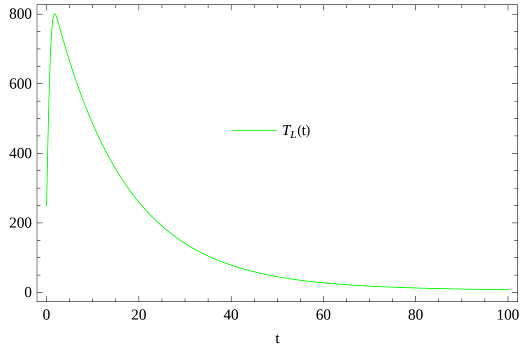

Figure 6. NMG4 solutions of $T_L(t)$ for the time interval (0,100) with the step size $\frac{1}{10}$ for the equation system (14-17)

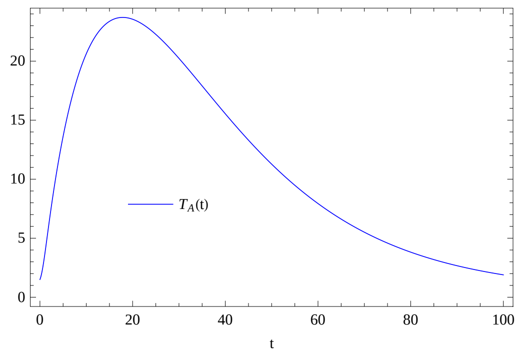

Figure 7. NMG4 solutions of $T_A(t)$ for the time interval (0,100) with the step size $\frac{1}{10}$ for the equation system (14-17)

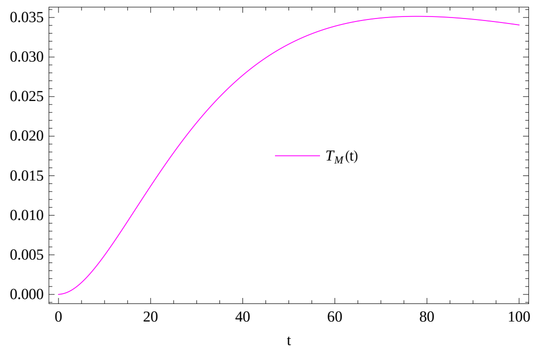

Figure 8. NMG4 solutions of $T_M(t)$ for the time interval (0,100) with the step size $\frac{1}{10}$ for the equation system (14-17)

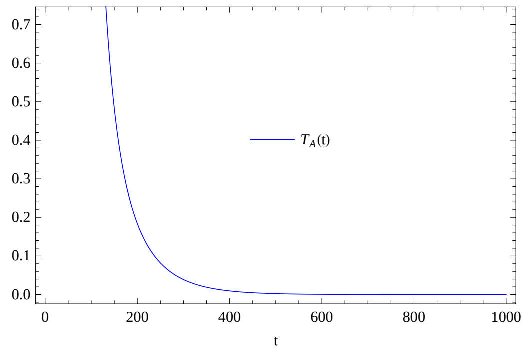

Figure 9. NMG4 solutions of $T_A(t)$ for the time interval (0,1000) with the step size $\frac{1}{10}$ for the equation system (14-17)

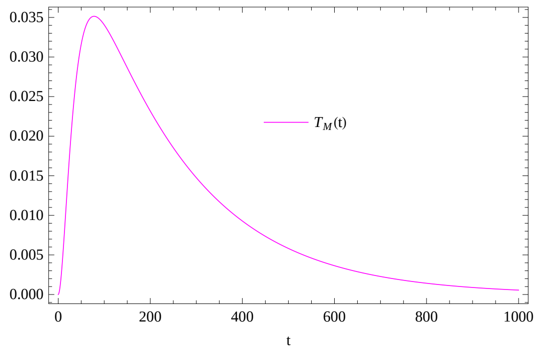

Figure 10. NMG4 solutions of $T_M(t)$ for the time interval (0,1000) with the step size $\frac{1}{10}$ for the equation system (14-17)

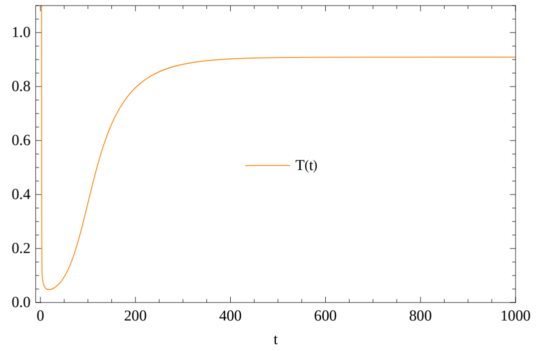

Figure 11. NMG4 solutions of $T(t)$ for the time interval (0,1000) with the step size $\frac{1}{10}$ for the equation system (14-17)

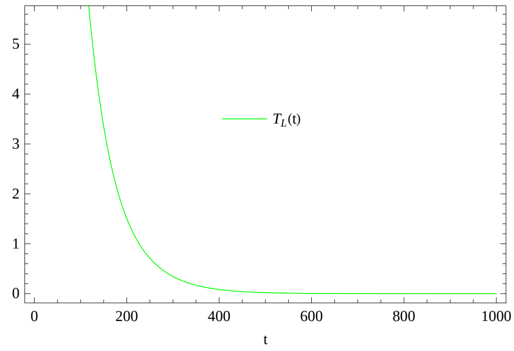

Figure 12. NMG4 solutions of $T_L(t)$ for the time interval (0,1000) with the step size $\frac{1}{10}$ for the equation system (14-17)

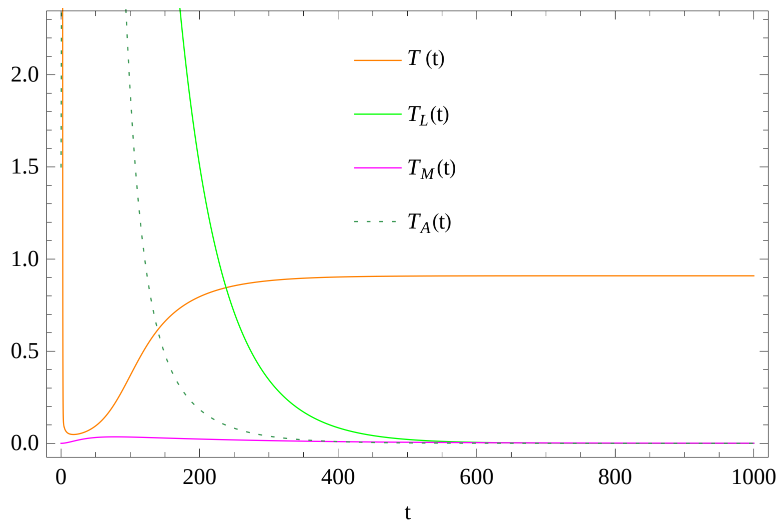

Figure 13. NMG4 solutions of $T(t), T_L(t), T_A(t)$ and $T_M(t)$ for the time interval (0,1000) with the step size $\frac{1}{10}$ for the equation system (14-17)

4 Conclusion

In this study, a reliable, efficient and structure preserving [17] numerical algorithm based on the Magnus series expansion is applied to solve the nonlinear differential equation system (14-17) which occurs in Human T-Cell Lymphotropic Virus I (HTLV-I) infection of $CD4^+$ T-cells model. The obtained results by NMG4 and the fourth order explicit RungeâĂŞKutta (RK45) are compared in Table 1-4. The obtained results are shown in Fig. 1-13. The numerical solutions of the NMG4 are in excellent agreement with respect to the RK45 solutions. As a result Magnus Expansion Method is an efficient and accurate tool for Human T-Cell Lymphotropic Virus I (HTLV-I) infection of $CD4^+$ T-cells model.

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