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Cosmology without Dark Matter

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Abstract We continue the study of the nonstandard model for the Universe in which we take a vacuum solution of Einstein's equations as background and treat visible matter and radiation as perturbations. Dark matter and dark energy are not introduced. The metric perturbations are obtained from the non-diagonal elements of the perturbed Einstein's equations. Using these results in the diagonal elements yields the energy momentum tensor of the Universe. We find a finite matter density, two pressure components p_r and p_ϑ vanish, but the longitudinal pressure p_ϕ is nonzero. The exceptional case of spherical perturbation leads to the isotropic cosmic microwave background. The same technique can also be used to describe galaxies with arbitrary rotation curves.

Keywords: Cosmology

1 Introduction

Our nonstandard cosmology is based on the following observational facts: (i) the isotropy of the cosmic microwave background and (ii) the smallness of the density of visible matter and the absence of dark matter in direct searches. The first implies that the cosmic gravitational field is spherically symmetric. According to (ii) we neglect sources in the zeroth approximation and introduce them by first order perturbation theory. So we start from a spherically symmetric vacuum solution of Einstein's equation. By Birkhoff's theorem this must be the inner Schwarzschild solution in Lemaître coordinates with a line element of the form [1]

$$ds^2 = dt^2 - X(t)^2 dr^2 - R(t)^2(d\vartheta^2 + \sin^2\vartheta d\phi^2). \quad (1.1)$$

The same solution is singled out of the much larger class of spherically symmetric dust solutions (LTB model) by a theoretical argument [2]. The time-dependent functions $R(t)$ and $Y(t)$ have been determined in such a way that the measured Hubble diagram is reproduced for redshifts between $0 < z < 10$ [3].

The next step is the analysis of perturbations of the background (1.1). This was started in [3] by using the results of Zerilli on the perturbation of the outer Schwazschild solution [4]. The first order of perturbation theory for spherical harmonics order $l \geq 1$ leads to a finite matter density, but the pressure remains zero (dust). The spherically symmetric perturbation $l = 0$ gives CMB as shown in a later paper. So nonstandard cosmology provides a natural explanation why radiation is isotropic but matter is not. Because of the importance of this perturbation theory we have decided to reconsider the problem from scratch again. Since the library of Princeton University does not deliver a copy of Zerilli's PhD thesis and in [4] no details are given, we have to repeat all calculations carefully. This is also necessary for the further applications.

In nonstandard cosmology the energy-momentum tensor $t_{\mu\nu}$ in Einstein's equation is obtained as follows. In comoving coordinates (1.1) this tensor has diagonal elements only. This can be taken as the definition of the comoving coordinates. Then setting the non-diagonal elements of the perturbed Einstein equations equal to zero gives the perturbations $h_{\mu\nu}$ of the metric. Using these results in the diagonal elements yields the tensor $t_{\mu\nu}$. This method does not work for spherical perturbations $l = 0$. This case is briefly described at the end of the paper, the details are given in a later article.

The paper is organized as follows. In the next section we give a synopsis of the perturbation calculation. In sect.3 the evolution equations for the metric perturbations are derived. In sect.4 the energy-momentum tensor is calculated, in particular the energy density. In the last section we discuss the application of the results to observations and to further studies.

Note added in proof:

The reader may ask: What about the nontrivial rotation curves in galaxies which have been the first indication of dark matter ? This problem can be beautifully solved by applying the technique of this paper to the *outer* Schwarzschild solution in suitable coordinates. So nonstandard cosmology fully agrees with observations. But there is a price to be paid: We must give up the Copernican Principle of spatial homogeneity[8]. Instead we favour the Ptolemaic Principle which says that we really live near a preferred place in the Universe, namely the center of spherical symmetry of CMB.

2 The Nonstandard Background and Its Perturbations

We work with comoving coordinates t, r, ϑ, ϕ and choose units so that $c = 1 = G$. The components of the nonstandard metric tensor are equal to

$$g_{00} = 1, \quad g_{11} = -X^2(t), \quad g_{22} = -R^2(t), \quad g_{33} = -R^2(t)\sin^2\vartheta, \tag{2.1}$$

and zero otherwise. The non-vanishing Christoffel symbols are given by

$$\Gamma^0_{11} = \dot{X}X, \quad \Gamma^0_{22} = \dot{R}R, \quad \Gamma^0_{33} = \dot{R}R\sin^2\vartheta \tag{2.2}$$

$$\Gamma^1_{01} = \frac{\dot{X}}{X}, \quad \Gamma^2_{02} = \frac{\dot{R}}{R} = \Gamma^3_{03} \tag{2.3}$$

$$\Gamma^2_{33} = -\sin\vartheta\cos\vartheta, \quad \Gamma^3_{23} = \cot\vartheta. \tag{2.4}$$

Here the dot always stands for $\partial/\partial t$. We also need the Riemann curvature tensor which has the following non-vanishing components

$$R^0_{101} = \ddot{X}X, \quad R^0_{202} = \ddot{R}R, \quad R^0_{303} = \ddot{R}R\sin^2\vartheta \tag{2.5}$$

$$R^1_{212} = \dot{R}R\frac{\dot{X}}{X}, \quad R^1_{313} = \dot{R}R\frac{\dot{X}}{X}\sin^2\vartheta \quad R^3_{232} = 1 + \dot{R}^2. \tag{2.6}$$

The time dependence of $X(t)$ and $R(t)$ is most easily given in parametric form

$$t = T_L(w - \sin w \cos w) \tag{2.7}$$

$$X(t) = |\cot w|, \quad R(t) = T_L \sin^2 w, \tag{2.8}$$

where T_L is of the order of the Hubble time. The Big Bang corresponds to $w = \pi/2$ and $w = \pi$ gives the end of the comoving time when the Universe is infinitely expanded [2]. In the following most calculations are done without using this time dependence, but the relations

$$\dot{R}(t) = X(t) \quad X^2 = \frac{T_L}{R} - 1 \tag{2.9}$$

will sometimes be used.

It is our aim to solve the linear perturbation equation of Einstein's equation which is of the form

$$\delta G_{\mu\nu}(g_{\alpha\beta})(h_{\varrho\sigma}) = 8\pi t_{\mu\nu}. \tag{2.10}$$

Here $h_{\varrho\sigma}$ are the metric perturbations of the nonstandard background to be calculated. Since the background is a vacuum solution we have written no delta on the right-hand side, that means the total energy-momentum tensor must come out from (2.10). It is well known that the linear perturbation of the Ricci tensor is obtained by covariant differentiation

$$\delta R_{\mu\nu} = -\frac{1}{2}\Big(\nabla^\alpha\nabla_\alpha h_{\mu\nu} - \nabla^\alpha\nabla_\nu h_{\mu\alpha} - \nabla^\alpha\nabla_\mu h_{\nu\alpha} + \nabla_\nu\nabla_\mu h^\alpha_\alpha\Big). \tag{2.11}$$

Then the left-hand side of (2.10) becomes

$$\delta R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Big(\nabla^\beta\nabla^\alpha h_{\alpha\beta} - \nabla_\beta\nabla^\beta h^\alpha_\alpha - h_{\alpha\beta}R^{\alpha\beta}\Big) - \frac{1}{2}h_{\mu\nu}R.$$

Following Zerilli [4] we commute the covariant derivatives by means of the curvature tensor which gives

$$2\delta G_{\mu\nu} = -\nabla^\alpha\nabla_\alpha h_{\mu\nu} + \nabla_\nu f_\mu + \nabla_\mu f_\nu - 2R^\beta_{\nu\alpha\nu}h^\alpha_\beta - \nabla_\nu\nabla_\mu h^\alpha_\alpha + R^\beta_\nu h_{\mu\beta} + R^\beta_\mu h_{\nu\beta} +$$
$$-g_{\mu\nu}(\nabla^\beta f_\beta - \nabla^\beta\nabla_\beta h^\alpha_\alpha) - Rh_{\mu\nu} + g_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta}$$

where

$$f_\mu = \nabla^\alpha h_{\mu\alpha}. \tag{2.12}$$

This greatly simplifies the calculations because the Ricci tensor vanishes since the background is a vacuum solution. There remains to calculate

$$2\delta G_{\mu\nu} = -\nabla^\alpha\nabla_\alpha h_{\mu\nu} + \nabla_\nu f_\mu + \nabla_\mu f_\nu - 2R^\beta_{\nu\alpha\nu}h^\alpha_\beta - \nabla_\nu\nabla_\mu h^\alpha_\alpha +$$
$$-g_{\mu\nu}(\nabla^\alpha f_\alpha - \nabla^\beta\nabla_\beta h^\alpha_\alpha). \tag{2.13}$$

It is necessary to choose a gauge. There is a preferred gauge due to Regge and Wheeler [6] where $h_{\mu\nu}$ is of the form

$$h_{\mu\nu} = \begin{pmatrix} -H_2 & XH_1 & 0 & 0 \\ XH_1 & -X^2H_0 & 0 & 0 \\ 0 & 0 & R^2K & 0 \\ 0 & 0 & 0 & R^2K\sin^2\vartheta \end{pmatrix} Y_l^m(\vartheta.\phi). \tag{2.14}$$

Here Y_l^m denote the spherical harmonics, the functions H_0, H_1, H_2 and K depend on t and r only. Our notation is such that we can compare our results with Zerilli's. It is known that the calculation in the Regge-Wheeler gauge is equivalent with the use of gauge invariant quantities [7]. Nevertheless, as we shall see, it is highly nontrivial that the angular dependence can be separated by means of the ansatz (2.14). Using the Christoffel symbols (2.2-4) all covariant derivatives can now be calculated in terms of the functions in (2.14). First we calculate the components of the vector f_μ (2.12):

$$f_0 = \Big[-\partial_0 H_2 - \frac{1}{X}\partial_1 H_1 - \frac{\dot X}{X}H_0 - \Big(\frac{\dot X}{X} + 2\frac{\dot R}{R}\Big)H_2 + 2\frac{\dot R}{R}K\Big]Y$$
$$f_1 = \Big[X\partial_0 H_1 + \partial_1 H_0 + 2\Big(\dot X + X\frac{\dot R}{R}\Big)H_1\Big]Y \tag{2.15}$$
$$f_2 = -K\partial_2 Y, \quad f_3 = -K\partial_3 Y.$$

Next in calculating $\nabla^\alpha f_\alpha$ angular derivatives appear which operate on the spherical harmonics. All second derivatives combine to the square of the angular momentum operator which satisfies

$$\partial_2^2 Y_l^m + \cot\vartheta\partial_2 Y_l^m + \frac{1}{\sin^2\vartheta}\partial_3^2 Y_l^m = -l(l+1)Y_l^m. \tag{2.16}$$

Details are given in the Appendix.

3 Evolution of the Metric Perturbations

In this section we calculate the non-diagonal elements of the Einstein tensor and put them equal to zero. The simplest computation is δG_{23}. Here only the terms

$$-\nabla_3\nabla_2 h^\alpha_\alpha = (-\partial_3\partial_2 + \cot\vartheta\partial_3)(H_0 - H_2 - 2K)Y \tag{3.1}$$

and $\nabla_3 f_2$, $\nabla_2 f_3$ contribute in (2.13), so that we obtain

$$2\delta G_{23} = (\cot\vartheta\partial_3 - \partial_3\partial_2)(H_0 - H_2)Y = 0.$$

This yields the first relation

$$H_0(t,r) = H_2(t,r) \tag{3.2}$$

which has to be used at various places in the following in order to separate the angular dependence.

Next we turn to δG_{02}. Here we need

$$\nabla^\alpha\nabla_\alpha h_{02}=2\frac{\dot R}{R}(K-H_2)\partial_2 Y \tag{3.3}$$

and

$$\nabla_2\nabla_0 h^\alpha_\alpha=\Big(\partial_0-\frac{\dot R}{R}\Big)(H_0-H_2-2K)\partial_2 Y.$$

Then we get

$$2\delta G_{02}=\Big[\partial_0(K-H_0)-\frac{1}{X}\partial_1 H_1+H_0\Big(\frac{\dot R}{R}-\frac{\dot X}{X}\Big)-H_2\Big(\frac{\dot R}{R}+\frac{\dot X}{X}\Big)\Big]\partial_2 Y=0 \tag{3.4}$$

The same equation is obtained from δG_{03}. This is the first of the three first order linear partial differential equations for the metric perturbations H_1, H_2, K which we are going to derive. However, this equation holds for $l\geq 1$ only. The reason is that $\partial_2 Y_0^0=0$. The same conclusion is true for (3.2), too. The case $l=0$ requires a special treatment, so we assume $l>0$ from now on. We shall return to this point at the end of the paper.

To compare our results with Zerilli's we must introduce the Schwarzschild variable R instead of time t. This is easily done by the substitutions

$$\partial_0=X\partial_R,\quad \dot R=X,\quad \dot X=-\frac{T_L}{2R^2} \tag{3.5}$$

and

$$X^2=\frac{T_L}{R}-1. \tag{3.6}$$

Then multiplying (3.4) by X we find

$$\Big(\frac{T_L}{R}-1\Big)\partial_R(K-H_0)-\partial_1 H_1+H_0\Big(\frac{3T_L}{2R^2}-\frac{1}{R}\Big)+H_2\Big(\frac{1}{R}-\frac{T_L}{2R^2}\Big)=0. \tag{3.7}$$

This agrees with Zerilli's equation (C7e), remembering that $T_L=2m$ in the Schwarzschild solution.

The next non-diagonal element is δG_{12}. Here we have

$$\nabla^\alpha\nabla_\alpha h_{12}=2\frac{\dot R}{R}XH_1\partial_2 Y$$

so that

$$2\delta G_{12}=[\partial_1(K+H_2)+X\partial_0 H_1+2\dot X H_1]\partial_2 Y=0. \tag{3.8}$$

After substituting (3.5-6) this is Zerilli's equation (C7d)

$$\Big(\frac{T_L}{R}-1\Big)\partial_R H_1+\partial_1(K+H_2)-\frac{T_L}{R^2}H_1=0= \tag{3.9}$$

$$=\partial_R\Big[\Big(\frac{T_L}{R}-1\Big)H_1\Big]+\partial_1(K+H_2). \tag{3.10}$$

From δG_{13} we get the same equation.

The most complicated non-diagonal element is δG_{01}. Here second derivatives appear, for example

$$\Box h_{01}=g^{\alpha\beta}\partial_\alpha\partial_\beta h_{01}=$$

$$=\Big(\partial_0^2-\frac{1}{X^2}\partial_1^2+\frac{l(l+1)}{R^2}+\frac{\cot\vartheta}{R^2}\partial_2\Big)h_{01} \tag{3.11}$$

where (2.16) has been used. The angular depending term with $\cot\vartheta$ must cancel. The details are shown in the appendix, where we derive the result

$$\nabla_\alpha\nabla^\alpha h_{01}=X\partial_0^2 H_1+\Big(\dot X+2\frac{\dot R}{R}X\Big)\partial_0 H_1-\frac{1}{X}\partial_1^2 H_1-$$

$$-2\frac{\dot{X}}{X}\partial_1(H_0+H_2)+\frac{X}{R^2}l(l+1)H_1-\Big(4\frac{\dot{X}^2}{X}+2\frac{\dot{R}^2}{R^2}X\Big)H_1. \tag{3.12}$$

The expressions for $\nabla_1 f_0$ and $\nabla_0 f_1$ are also given in the appendix. We get a contribution from the Riemann tensor

$$R^1_{001}=\frac{\ddot{X}}{X}$$

which is multiplied by $-2h^0_1=-2XH_1$. The final result is

$$\frac{\delta G_{01}}{Y}=\partial_0\partial_1 K-\frac{\dot{R}}{R}\partial_1 H_2+\Big(\frac{\dot{R}}{R}-\frac{\dot{X}}{X}\Big)\partial_1 K-X\frac{l(l+1)}{2R^2}H_1=0. \tag{3.13}$$

With the substitutions (3.5-6) we have

$$\partial_R\partial_1 K-\frac{1}{R}\partial_1 H_2+\Big(\frac{1}{R}+\frac{T_L}{2R(T_L-R)}\Big)\partial_1 K-\frac{l(l+1)}{2R^2}H_1=0. \tag{3.14}$$

This is Zerilli's equation (C7b).

Taking (3.2) into account, we have now obtained the three partial differential equations (3.7), (3.9) and (3.14) for the 3 unknown functions H_1, H_2 an K of (t,r). Since the r-dependence only appears in derivatives ∂_1 we go over to Fourier transformed quantities

$$\hat{f}(R,q)=(2\pi)^{-1/2}\int f(R,r)e^{iqr}dr. \tag{3.15}$$

Then the derivative ∂_1 goes over to a factor $-iq$ and we omit the hat in the following. Instead of partial differential equations we now have the following 3 ordinary differential equations in time or R:

$$\partial_R K=\frac{H_2}{R}-\frac{3T_L-2R}{2R(T_L-R)}K-\frac{l(l+1)}{2R^2}H_3 \tag{3.16}$$

$$\partial_R H_2=\frac{2T_L-R}{R(T_L-R)}H_2-\frac{3T_L-2R}{2R(T_L-R)}K+\Big(\frac{q^2R}{R-T_L}-\frac{l(l+1)}{2R^2}\Big)H_3 \tag{3.17}$$

$$\partial_R H_3=\frac{R}{T_L-R}(K+H_2)+\frac{T_L}{R(T_L-R)}H_3. \tag{3.18}$$

Here we have introduced the function

$$H_3=\frac{H_1}{iq} \tag{3.19}$$

in order to have pure real equations. These equations govern the time evolution of the metric perturbations and are further discussed in the last section.

4 Determination of the Energy-Momentum Tensor

Now we consider the diagonal elements of the perturbed Einstein's equation (2.10). Since the metric perturbations are fixed by the results of the last section, the diagonal elements determine the (diagonal) tensor

$$t_{\mu\nu}=\mathrm{diag}(\varrho,-g_{11}p_1,-g_{22}p_2,-g_{33}p_3) \tag{4.1}$$

on the right-hand side of (2.10). We start with δG_{11}. The separate pieces are given in the appendix. The final result

$$\frac{\delta G_{11}}{X^2Y}=\Big(\frac{T_L}{R}-1\Big)\partial_R^2K+\Big[\frac{3}{R}\Big(\frac{T_L}{R}-1\Big)-\frac{T_L}{2R^2}\Big]\partial_R K-$$
$$-\Big(\frac{T_L}{R}-1\Big)\frac{1}{R}\partial_R H_2+\frac{1}{R^2}(H_2-K) \tag{4.2}$$

agrees with Zerilli's equation (C7a). By differentiating (3.16) with respect to R and then substituting (3.16) and (3.17-18) a second time into (4.2) we get zero. That means the radial pressure $t_{11}=-g_{11}p_r$ vanishes.

The next diagonal element δG_{22} is more complicated. The calculation of $\nabla_\alpha\nabla^\alpha h_{22}$ is given by (A.7) in the appendix. Next we need

$$\nabla_2 f_2 = \partial_2 f_2 - \Gamma^0_{22} f_0 = -K\partial_2^2 Y +$$
$$+\dot R R\Big[\partial_0 H_2 + \frac{1}{X}\partial_1 H_1 + \frac{\dot X}{X}(H_0+H_2) - 2\frac{\dot R}{R}(K-H_2)\Big]Y. \tag{4.3}$$

This gets multiplied by 2 in (2.12). From the Riemann tensor we get three contributions

$$-2R^0_{202}h^0_0 = 2\ddot R R H_2 Y,\quad -2R^1_{212}h^1_1 = -2R\dot R\frac{\dot X}{X}H_0 Y$$
$$-2R^3_{232}h^3_3 = 2(1+\dot R^2)KY. \tag{4.4}$$

The large last contribution in (2.13) with $-g_{22}$ is taken from (A.4). Putting everything together we finally obtain

$$\frac{2\delta G_{22}}{R^2 Y} = \partial_0^2 K - \frac{1}{X^2}\partial_1^2 K - \partial_0^2 H_0 - \frac{1}{X^2}\partial_1^2 H_2 - \frac{2}{X}\partial_0\partial_1 H_1 +$$
$$+\partial_0 K\Big(2\frac{\dot R}{R}+\frac{\dot X}{X}\Big) - \partial_0 H_2\Big(\frac{\dot R}{R}+\frac{\dot X}{X}\Big) - 2\partial_1 H_1\Big(\frac{\dot R}{RX}+\frac{\dot X}{X^2}\Big) -$$
$$-\partial_0 H_0\Big(\frac{\dot R}{R}+2\frac{\dot X}{X}\Big) - (H_2+H_0)\Big(\frac{\ddot X}{X}+2\frac{\dot X\dot R}{XR}\Big) + \frac{l(l+1)}{R^2}(H_2-H_0) +$$
$$+2K\Big(\frac{1}{R^2}+\frac{\ddot R}{R}+\frac{\dot X\dot R}{XR}\Big). \tag{4.5}$$

After substituting ∂_0 by ∂_R we obtain Zerilli's equation (C7f) because the last term in (4.5) vanishes by (2.9):

$$\Big(\frac{T_L}{R}-1\Big)\partial_R^2(K-H_0) - \frac{R}{T_L-R}\partial_1^2(K+H_2) - 2\partial_R\partial_1 H_1 +$$
$$+\frac{2}{R}\Big(\frac{T_L}{2R}-1\Big)\partial_R K + \Big(\frac{1}{R}+\frac{T_L}{2R^2}\Big)\partial_R H_0 + \Big(\frac{1}{R}-\frac{T_L}{2R^2}\Big)\partial_R H_2 +$$
$$+\frac{2}{T_L-R}\Big(1-\frac{T_L}{2R}\Big)\partial_1 H_1 = \frac{\delta G_{22}}{R^2 Y}. \tag{4.6}$$

Again we substitute all derivatives by the evolution equations (3.16-18). Since we get zero again, the second pressure components t_{22} also vanishes.

In δG_{33} the angular dependence is most complicated. Zerilli probably did not calculate it because we get some new results. We have

$$\nabla_\alpha\nabla^\alpha h_{33} = \sin^2\vartheta\Big(R^2\partial_0^2 K + (2\dot R R + R^2\frac{\dot X}{X})\partial_0 K - \frac{R^2}{X^2}\partial_1^2 K\Big)Y +$$
$$+K\Big[\sin^2\vartheta\Big(l(l+1)-2\dot R^2\Big) + 4\cos^2\vartheta - 2 + 4\sin\vartheta\cos\vartheta\frac{\partial_2 Y}{Y}\Big]Y +$$
$$+2H_2\dot R^2\sin^2\vartheta Y. \tag{4.7}$$

In

$$\nabla_3 f_3 = -\partial_3^2 KY - \Gamma^\alpha_{33} f_\alpha \tag{4.8}$$

there is even a break of axial symmetry due to the relation

$$\partial_3^2 Y_l^m(\vartheta,\phi) = -m^2 Y_l^m. \tag{4.9}$$

We find

$$\nabla_3 f_3 = m^2 KY - \sin\vartheta\cos\vartheta K\partial_2 Y -$$
$$+\dot R R\sin^2\vartheta\Big(\partial_0 H_2 + \frac{1}{X}\partial_1 H_1 + \frac{\dot X}{X}H_0 + (\frac{\dot X}{X}+2\frac{\dot R}{R})H_2 - 2\frac{\dot R}{R}K\Big)Y. \tag{4.10}$$

There are three terms from the Riemann tensor R^0_{303}, R^1_{313} and R^2_{323} and

$$\nabla_3\nabla_3 h^\alpha_\alpha = \partial_3^2 h^\alpha_\alpha - \Gamma^\alpha_{33}\partial_\alpha h^\beta_\beta = -m^2(H_0 - H_2 - 2K)Y -$$

$$-\dot{R}R\sin^2\vartheta\partial_0(H_0 - H_2 - 2K)Y + \sin\vartheta\cos\vartheta(H_0 - H_2 - 2K)\partial_2 Y. \tag{4.11}$$

The last big piece in (2.13) follows from (A.4) in the appendix.

The final result is

$$\frac{2\delta G_{33}}{R^2\sin^2\vartheta Y} = \partial_0^2(K - H_0) - \frac{1}{X^2}\partial_1^2(K + H_2) - \frac{2}{X}\partial_0\partial_1 H_1 +$$

$$+\partial_0 K\Big(2\frac{\dot{R}}{R} + \frac{\dot{X}}{X}\Big) - \partial_0 H_2\Big(\frac{\dot{R}}{R} + \frac{\dot{X}}{X}\Big) - \partial_0 H_0\Big(\frac{\dot{R}}{R} + 2\frac{\dot{X}}{X}\Big) - 2\partial_1 H_1\Big(\frac{\dot{R}}{RX} + \frac{\dot{X}}{X^2}\Big) +$$

$$+2K\Big[\frac{1}{R^2} + \frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + \frac{\dot{X}}{X}\frac{\dot{R}}{R}\Big] + \frac{2K}{R^2\sin^2\vartheta}\Big(1 - 2\cos^2\vartheta - 2\sin\vartheta\cos\vartheta\frac{\partial_2 Y}{Y}\Big). \tag{4.12}$$

As far as the terms with derivatives are concerned, this agrees with (4.5); we have left out the terms with $H_0 = H_2$ which have zero factors after substituting (3.5-6). It follows from the previous analysis of δG_{22} that these derivative terms vanish if the evolution equations (3.16-18) are taken into account. Then using (3.5-6) we conclude

$$\delta G_{33} = K\Big[\dot{R}^2\sin^2\vartheta Y + (1 - 2\cos^2\vartheta)Y - 2\sin\vartheta\cos\vartheta\partial_2 Y\Big]. \tag{4.13}$$

This gives a non-zero pressure $p_3 = p_\phi$. It also shows that the separation of the angular variables is no longer complete. The two last pressure terms come from the square bracket in (4.7) and are not compensated by other contributions.

The most interesting diagonal element is δG_{00} which determines the matter density ϱ (4.1). The long calculation of $\nabla_\alpha\nabla^\alpha h_{00}$ is again shown in the appendix (A.6). Since there is no Christoffel symbol with Γ_{00} we simply have

$$\nabla_0 f_0 = \partial_0 f_0$$

and

$$\nabla_0\nabla_0 h^\alpha_\alpha = \partial_0^2(H_0 - H_2 - 2K).$$

The Riemann tensor gives contributions from R^1_{010}, R^2_{020} and R^3_{030}. The last big piece is taken from (A.4) again. Collecting all terms we arrive at

$$\frac{2\delta G_{00}}{Y} = \frac{2}{X^2}\partial_1^2 K + 4\frac{\dot{R}}{RX}\partial_1 H_1 + 2\frac{\dot{R}}{R}\partial_0 H_0 - 2\Big(\frac{\dot{R}}{R} + \frac{\dot{X}}{X}\Big)\partial_0 K +$$

$$+H_2\Big(2\frac{\dot{R}^2}{R^2} - 2\frac{\ddot{R}}{R} - \frac{\ddot{X}}{X} + 4\frac{\dot{X}\dot{R}}{XR}\Big) + H_0\Big(\frac{\ddot{X}}{X} + 2\frac{\dot{X}\dot{R}}{XR}\Big) -$$

$$-2K\Big(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{\dot{X}\dot{R}}{XR}\Big) + \frac{l(l+1)}{R^2}(H_0 - K). \tag{4.14}$$

After substituting Schwarzschild variables

$$\Big(\frac{T_L}{R} - 1\Big)^{-1}\partial_1^2 K + \frac{2}{R}\partial_1 H_1 + \frac{1}{R}\Big(\frac{T_L}{R} - 1\Big)\partial_R(H_0 - K) +$$

$$+\frac{T_L}{2R^2}\partial_R K + \frac{1}{R^2}(K - H_2) + \frac{l(l+1)}{2R^2}(H_0 - K) = \frac{\delta G_{00}}{Y} \tag{4.15}$$

this agrees with Zerilli's equation (C7c).

According to (2.10) and (4.1) the result (4.15) gives the matter density $8\pi\varrho/Y$. After substituting ∂_1 by $-iq$ and eliminating the derivatives ∂_R with help of the evolution equations (3.16-18) we obtain

$$\frac{8\pi\varrho}{Y} = K\Big[\frac{1}{R^2} - \frac{l(l+1)}{2R^2} - \frac{3T_L}{4R^3} - \frac{q^2 R}{T_L - R} - \frac{T_L}{4R^2(T_L - R)}\Big] +$$

$$+H_2\Big[\frac{3T_L}{2R^3} - \frac{1}{R^2} + \frac{l(l+1)}{2R^2}\Big] + H_3\Big[\frac{q^2}{R} - \frac{T_L}{4R^4}l(l+1)\Big]. \tag{4.16}$$

Due to the l-dependence this is an anisotropic matter density which will be discussed in the next section.

5 Discussion

To apply the perturbative results to our Universe we must first integrate the evolution equations. This is best done by using the variable

$$x = \alpha(1+z) \tag{5.1}$$

which is directly related to the redshift z. The parameter α is determined by the Hubble diagram which gives the value $\alpha = 2.59$ [3]. The transformation of variables is given by

$$R = T_L \frac{x^2}{x^2+1}, \quad X = \frac{1}{x} \tag{5.2}$$

$$T_L - R = \frac{T_L}{x^2+1} \tag{5.3}$$

and

$$\frac{d}{dR} = \frac{(x^2+1)^2}{2T_L x}\frac{d}{dx}. \tag{5.4}$$

In the new variable x our the evolution equations read

$$K' = -\frac{x}{x^2+1}\Big(1+\frac{3}{x^2}\Big)K + \frac{2H_2}{x(x^2+1)} - \frac{l(l+1)}{T_L x^3}H_3 \tag{5.5}$$

$$H_2' = -\frac{x}{x^2+1}\Big(1+\frac{3}{x^2}\Big)K + \frac{2}{x^2+1}\Big(x+\frac{2}{x}\Big)H_2 - \\ -\Big[q^2T_L^2\frac{2x^3}{(x^2+1)^2} + \frac{l(l+1)}{x^3}\Big]\frac{H_3}{T_L} \tag{5.6}$$

and

$$H_3' = \frac{2T_L x^3}{(x^2+1)^2}(K+H_2) + \frac{2}{x}H_3. \tag{5.7}$$

Here the prime is the derivative with respect to x.

The system of 3 linear first order ode's can be numerically integrated by any ode-solver, for example DSolve of Mathematica. For not to small x a power series expansion can be used. It is of the form

$$K = \frac{a_1}{x} + \frac{a_2}{x^3} + \frac{a_3}{x^5} + \dots \tag{5.8}$$

$$H_2 = \frac{b_1}{x} + \frac{b_2}{x^3} + \dots \quad H_4 = \frac{H_3}{T_L} = \frac{c_1}{x} + \frac{c_2}{x^3} + \dots \tag{5.9}$$

Here all coefficients are determined by a_1 which fixes the overall normalization of the metric perturbations. A stable numerical integration of the evolution equations must go from large to small redshift. So one starts at $z = 10$, say, with normalization $a_1 = 1$ calculating the initial values by the power series (5.8-9). Then one integrates down to the present $z = 0$ where one can compare the result with measured matter density (4.16). This then gives the correct normalization. It turns out that the numerical solution of the evolution equations is only necessary for the late Universe $z < 10$, for $z > 10$ the power series (5.8-9) can be used if l is not too big.

For the approach to the Big Bang $z = \infty$ the power series are perfect. The metric perturbation go to zero as $1/x$. However the energy density (4.16) grows proportional to x. In fact we have shown in [3] eq.(5.3) that

$$\frac{\varrho}{\varrho_{\rm crit}} = \frac{2}{3}\frac{\alpha^6}{(\alpha^2+1)^4}\frac{(x^2+1)^2}{x^3}\Big)|a_1|\Big[Q^2 + \frac{1}{2} + O(x^{-2})\Big] \tag{5.10}$$

where $\varrho_{\rm crit}$ is the critical density. It is a nice feature of nonstandard cosmology that the Big Bang is not a singularity of Einstein's equation, because it corresponds to the horizon in the Schwarzschild solution. But the growth of the energy density (5.10) indicates that first order perturbation theory cannot be the whole story. We shall treat second order perturbation theory in another paper. The longitudinal pressure

p_ϕ obtained from (4.13) is proportional to $K = O(x^{-1})$. So it is small compared to the density $\varrho = O(x)$ for large redshift.

It is important to remember that we have excluded spherical perturbations $l = 0$. This requires a separate treatment which goes as follows. Instead of three equations for the metric perturbations in sect.4 we have only one. Then an equation of state is needed to solve the perturbed Einstein's equations. By solving Maxwell equations in the nonstandard background we find that the $l = 0$ perturbation comes from incoherent isotropic radiation. This is CMB which will be discussed in a forthcoming paper. The rotation curves in galaxies can be understand without dark matter by a modification of the outer Schwarzschild solution. But that is another story.

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Appendix

The computation of the very many covariant derivatives is cumbersome. Here is an example:

$$\nabla^\alpha f_\alpha = g^{\alpha\beta}\nabla_\beta f_\alpha = \partial_0 f_0 - \frac{1}{X^2}(\partial_1 f_1 - \Gamma^0_{11} f_0) -$$

$$-\frac{1}{R^2}(\partial_2 f_2 - \Gamma^0_{22} f_0) - \frac{1}{R^2\sin^2\vartheta}(\partial_3 f_3 - \Gamma^\alpha_{33} f_\alpha).$$

Here in the last term two Christoffel symbols $\alpha = 0, 2$ contribute. Substituting the explicit expressions (3.19) and using (3.16) we finally obtain

$$\frac{\nabla^\alpha f_\alpha}{Y} = -\partial_0^2 H_2 - \frac{1}{X^2}\partial_1^2 H_0 - \frac{2}{X}\partial_0\partial_1 H_1 - \Big(2\frac{\dot X^2}{X^2} + 4\frac{\dot R}{RX}\Big)\partial_1 H_1 -$$

$$-\Big(2\frac{\dot X}{X} + 4\frac{\dot R}{R}\Big)\partial_0 H_2 - \frac{\dot X}{X}\partial_0 H_0 + 2\frac{\dot R}{R}\partial_0 K +$$

$$-H_0\Big(\frac{\ddot X}{X} + 2\frac{\dot R\dot X}{RX}\Big) - H_2\Big(\frac{\ddot X}{X} + 2\frac{\ddot R}{R} + 2\frac{\dot R^2}{R^2} + 4\frac{\dot R\dot X}{RX}\Big) +$$

$$+2K\Big(\frac{\ddot R}{R} + \frac{\dot R^2}{R^2} + \frac{\dot R\dot X}{RX} - \frac{l(l+1)}{2R^2}\Big). \tag{A.1}$$

Another long calculation is

$$\nabla_\alpha\nabla^\alpha h_{01} = \Box h_{01} - 2\Gamma^1_{01}\partial^1 h_{11} - 2\Gamma^0_{11}\partial^1 h_{00} - 2\Gamma^1_{01}\partial^0 h_{01} +$$

$$-\Big(g^{11}\Gamma^0_{11}\partial_0 + g^{22}\Gamma^0_{22}\partial_0 + g^{33}\Gamma^0_{33}\partial_0 + g^{33}\Gamma^2_{33}\partial_2\Big)h_{01} +$$

$$+\Big(g^{11}\Gamma^1_{01}\Gamma^0_{11} + g^{22}\Gamma^2_{02}\Gamma^0_{22} + g^{33}\Gamma^3_{03}\Gamma^0_{33}\Big)h_{01} - h_{01}\partial_0\Gamma^1_{01} +$$

$$+\Big(g^{11}\Gamma^0_{11}\Gamma^1_{01} + g^{22}\Gamma^0_{22}\Gamma^1_{01} + g^{33}\Gamma^0_{33}\Gamma^1_{01}\Big)h_{01} +$$

$$+\Big(g^{00}\Gamma^1_{01}\Gamma^1_{01}+g^{11}\Gamma^0_{11}\Gamma^1_{01}\Big)h_{01}+2g^{11}\Gamma^1_{01}\Gamma^0_{11}h_{01}. \tag{A.2}$$

Here the term with Γ^2_{33} cancels the angular term with $\cot\vartheta$ in $\Box h_{01}$. After substituting all explicit expressions we arrive at the result (3.12).

For the diagonal elements we need the last term in (3.13):

$$\nabla^\beta\nabla_\beta h^\alpha_\alpha = g^{\mu\beta}[\partial_\beta\partial_\mu(H_0-H_2-2K)Y-\Gamma^\alpha_{\mu\beta}\partial_\alpha(H_0-H_2-2K)Y=$$
$$=\Box(H_0-H_2-2K)Y-\Big[-\Gamma^0_{11}\frac{1}{X^2}\partial_0-\Gamma^0_{22}\frac{1}{R^2}\partial_0-$$
$$-\Gamma^0_{33}\frac{1}{R^2\sin^2\vartheta}\partial_0-\Gamma^2_{33}\frac{1}{R^2\sin^2\vartheta}\partial_2\Big](H_0-H_2-2K)Y=$$
$$=\Big[\partial_0^2-\frac{1}{X^2}\partial_1^2+\frac{l(l+1)}{R^2}+\Big(\frac{\dot X}{X}+2\frac{\dot R}{R}\Big)\partial_0\Big](H_0-H_2-2K)Y. \tag{A.3}$$

Combining this with (A.1) gives

$$\frac{1}{Y}\Big(\nabla^\alpha f_\alpha-\nabla^\beta\nabla_\beta h^\alpha_\alpha\Big)=\partial_0^2(2K-H_0)-\frac{1}{X^2}\partial_1^2(2K-H_2)-\frac{2}{X}\partial_0\partial_1H_1-$$
$$-\Big(2\frac{\dot X}{X^2}+4\frac{\dot R}{XR}\Big)\partial_1H_1-\Big(2\frac{\dot X}{X}+2\frac{\dot R}{R}\Big)\partial_0H_0-\Big(\frac{\dot X}{X}+2\frac{\dot R}{R}\Big)\partial_0H_2+$$
$$+\Big(2\frac{\dot X}{X}+6\frac{\dot R}{R}\Big)\partial_0K-\Big(\frac{\ddot X}{X}+2\frac{\dot X\dot R}{XR}\Big)H_0-\Big(\frac{\ddot X}{X}+2\frac{\ddot R}{R}+2\frac{\dot R^2}{R^2}+4\frac{\dot X\dot R}{XR}\Big)H_2+$$
$$+\Big(2\frac{\ddot R}{R}+2\frac{\dot R^2}{R^2}+2\frac{\dot X\dot R}{XR}\Big)K+\frac{l(l+1)}{R^2}(H_2-H_0+K). \tag{A.4}$$

For δG_{11} we need

$$\nabla_\alpha\nabla^\alpha h_{11}=\Box h_{11}-4\Gamma^\varrho_{\alpha1}\partial^\alpha h_{\varrho1}-g^{\alpha\beta}\Gamma^\varrho_{\alpha\beta}\partial_\varrho h_{11}+$$
$$+2g^{\alpha\beta}h_{\varrho1}(-\partial_\beta\Gamma^\varrho_{1\alpha}+\Gamma^\sigma_{\alpha\beta}\Gamma^\varrho_{\sigma1}+\Gamma^\sigma_{1\beta}\Gamma^\varrho_{\alpha\sigma})+2g^{\alpha\beta}h_{\varrho\sigma}\Gamma^\varrho_{\beta1}\Gamma^\sigma_{\alpha1}$$
$$=\Big[-X^2\partial_0^2H_0+\partial_1^2H_0-\Big(X\dot X+2\frac{\dot R}{R}X^2\Big)\partial_0H_0+4\dot X\partial_1H_1+$$
$$+4\dot X\partial_1H_1+2\dot X^2H_2+H_0\Big(2\dot X^2-\frac{X^2}{R^2}l(l+1)\Big)\Big]Y \tag{A.5}$$

and

$$\nabla_1f_1=\partial_1f_1-\Gamma^0_{11}f_0=\Big[\partial_1\partial_0(XH_1)+\partial_1^2H_0+$$
$$+\Big(\dot X+2\frac{\dot RX}{R}\Big)\partial_1H_1+X\dot X\partial_0H_2+\dot X\partial_1H_1+\dot X^2(H_0+H_2)-$$
$$-2\dot XX\frac{\dot R}{R}(K-H_2)\Big]Y. \tag{A.6}$$

We get contributions from the components R^0_{101}, R^2_{121} and R^3_{131} of the Riemann tensor. In addition (A.4) must be used for the last term in (3.13).

Next we compute

$$\nabla_\alpha\nabla^\alpha h_{22}=\Box h_{22}-4\Gamma^2_{02}\partial^0h_{22}-\Big(g^{11}\Gamma^0_{11}\partial_0+g^{22}\Gamma^0_{22}\partial_0+$$
$$+g^{33}\Gamma^0_{33}\partial_0+g^{33}\Gamma^2_{33}\partial_2\Big)h_{22}-2h_{22}\partial_0\Gamma^2_{20}-2h_{22}\Big(\frac{\dot X}{X}+2\frac{\dot R}{R}\Big)\frac{\dot R}{R}+$$
$$+2h_{22}\Big(g^{00}\Gamma^2_{20}\Gamma^2_{02}+g^{22}\Gamma^0_{22}\Gamma^2_{02}+g^{33}\Gamma^3_{23}\Gamma^2_{33}\Big)+$$
$$+2\Big(h_{22}g^{00}(\Gamma^2_{02})^2+h_{00}g^{22}(\Gamma^0_{22})^2+h_{33}g^{33}(\Gamma^3_{32})^2\Big)$$

$$= \Big[R^2\partial_0^2 K - \frac{R^2}{X^2}\partial_1^2 K + \Big(2\dot{R}R + R^2\frac{\dot{X}}{X}\Big)\partial_0 K + 2\dot{R}^2 H_2 + [l(l+1) - 2\dot{R}^2]K\Big]Y. \qquad (A.7)$$

For δG_{00} we need

$$\nabla_\alpha\nabla^\alpha h_{00} = \Box h_{00} - 4\Gamma^1_{10}\partial^1 h_{10} - (g^{11}\Gamma^0_{11} + g^{22}\Gamma^0_{22} + g^{33}\Gamma^0_{33})\partial_0 h_{00} -$$

$$-g^{33}\Gamma^2_{33}\partial_2 h_{00} + 2h_{00}(g^{11}\Gamma^1_{01}\Gamma^0_{11} + g^{22}\Gamma^2_{02}\Gamma^0_{22} + g^{33}\Gamma^3_{03}\Gamma^0_{33}) +$$

$$+2\Big[h_{11}(\Gamma^1_{10})^2 g^{11} + h_{22}(\Gamma^2_{20})^2 g^{22} + h_{33}(\Gamma^3_{30})^2 g^{33}\Big] =$$

$$= \Big[-\Big(\partial_0^2 - \frac{1}{X^2}\partial_1^2 + \frac{l(l+1)}{R^2}\Big)H_2 + 4\frac{\dot{X}}{X^2}\partial_1 H_1 - \Big(\frac{\dot{X}}{X} + 2\frac{\dot{R}}{R}\Big)\partial_0 H_2 +$$

$$+H_2\Big(2\frac{\dot{X}^2}{X^2} + 4\frac{\dot{R}^2}{R^2}\Big) + 2\frac{\dot{X}^2}{X^2}H_0 - 4\frac{\dot{R}^2}{R^2}K\Big]Y. \qquad (A.8)$$

