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Polynomial Inequalities in Regions with Piecewise Asymptotically Conformal Curve in the Weighted Lebesgue Space

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Abstract In this present work, we study the Nikolskii type estimations for algebraic polynomials in the bounded regions with piecewise-asymptotically conformal curve, having interior and exterior zero angles, in the weighted Lebesgue space

Keywords: Algebraic polynomials, conformal mapping, assymptotically coformal curve, quasicircle

1 Introduction

Let $\mathbb{C}$ be a complex plane; $G \subset \mathbb{C}$ be a bounded region, with $0 \in G$ and Jordan boundary $L := \partial G$.

Let $\{\xi_j\}_{j=1}^m$ be a fixed system of distinct points on curve L located in the positive direction. For some finite region $G^* \subset \mathbb{C}$ such that $G \subset G^*$ and $z \in G^*$, consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows:

$$h(z) := \prod_{j=1}^{m} |z - \xi_j|^{\gamma_j}, \tag{1}$$

where $\gamma_j > -1$ for all $j = 1, 2, ..., m$.

For a rectifiable Jordan curve L and for $0 < p \leq \infty$, let $\mathcal{L}_p(h, L)$ denote the weighted Lebesgue space of complex-valued functions on L. Specifically, $f \in \mathcal{L}_p(h, L)$ if f is measurable and the following quasinorm (a norm for $1 \leq p \leq \infty$ and a $p-$norm for $0 < p < 1$) is finite:

$$\|f\|_p := \|f\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) |f(z)|^p |dz| \right)^{1/p}, \ 0 < p < \infty;$$
$$\|f\|_\infty := \|f\|_{\mathcal{L}_\infty(1,L)} := ess \sup_{z \in L} |f(z)|, \ p = \infty.$$

We denote by $\wp_n$, $n = 1, 2, ...$, the set of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

In this work, we study the following Nikolskii-type inequality

$$\|P_n\|_\infty \leq c_1 \mu_n(G, h, p) \|P_n\|_p, \tag{2}$$

for some general regions having interior and exterior zero angles of the power type, where $c_1 = c_1(G, p) > 0$ is a constant independent of n, h and P_n, and $\mu_n(G, h, p) \to \infty$, $n \to \infty$, depending on the geometrical properties of region G and weight function h in the neighborhood of the points $\{\xi_j\}_{j=1}^m$.

The first result of (2)-type, in case $h(z) \equiv 1$ and $L = \{z : |z| = 1\}$ for $0 < p < \infty$ was found by Jackson [18]. Another classical results similar to (2) belong to Szegö and Zigmund [31]. Suetin [32], [33] investigated this problem with sufficiently smooth Jordan curve. The estimate of (2)-type for $0 < p < \infty$ and $h(z) \equiv 1$ when L is a rectifiable Jordan curve was investigated by Mamedhanov [21], [22], Nikolskii [24, pp.122-133], Pritsker [29], Andrievskii [10, Theorem 6] and others. More references regarding the inequality of (2)-type, we can find in Milovanovic et al. [23, Sect.5.3].

Further, analogous estimates of (2) for some regions and the weight function $h(z)$ were obtained: in [2] ($p > 1$) and in [25] ($p > 0$, $h \equiv h_0$) for regions bounded by rectifiable quasiconformal curve having some general properties; in [4] ($p > 1$) for piecewise Dini-smooth curve with interior and exterior cusps; in [3] ($p > 1$) for regions bounded by piecewise smooth curve with exterior cusps but without interior cusps; in [5] ($p > 0$) for regions bounded by piecewise rectifiable quasiconformal curve with cusps; in [6] ($p > 0$) for regions bounded by piecewise quasismooth (by Lavrentiev) curve with cusps.

Now, let's give some definitions and notations.

Let z_1, z_2 be an arbitrary points on l and $l(z_1, z_2)$ denotes the subarc of l of shorter diameter with endpoints z_1 and z_2. The curve l is a quasicircle if and only if the quantity

$$\sup_{z_1,z_2\in l;\ z\in l(z_1,z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \tag{3}$$

is bounded [19, p.105]. Following to Lesley [20], we say that the curve l to be said "*$c-$quasiconformal*", if the quantity (3) bounded by positive constant c, independent from points z_1, z_2 and z. At the literature it is possible to find various functional definitions of the quasiconformal curves (see, for example, Def. 3, [26, pp.286-294], [19, p.105], [7, p.81], [27, p.107]).

The Jordan curve l is called asymptotically conformal ([12], [27]), if

$$\sup_{z_1,z_2\in l;\ z\in l(z_1,z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \to 1, \qquad |z_1 - z_2| \to 0. \tag{4}$$

We will denote this class as AC, and will write $G \in AC$, if $L := \partial G \in AC$.

The asymptotically conformal curves occupy a special place in the problems of the geometric theory of functions of a complex variable. These curves in various problems have been studied by J.M. Anderson, J. Becker and F.D. Lesley [8], E.M.Dyn'kin [13], Ch. Pommerenke, S.E. Warschawski [28], V.Ya. Gutlyanskii, V.I. Ryazanov [14], [15], [16] and others. According to the geometric criteria of quasiconformality of the curves ([7, p.81], [27, p.107]), every asymptotically conformal curve is a quasicircle. Every smooth curve is asymptotically conformal but corners are not allowed. It is well known that quasicircles can be non-rectifiable (see, for example, [11], [19, p.104]). The same is true for asymptotically conformal curves.

We say that $L \in \widetilde{AC}$, if $L \in AC$ and L is rectifiable. A Jordan arc ℓ is called asymptotically conformal arc, when ℓ is a part of some asymptotically conformal curve.

Now, we define a new class of regions bounded by piecewise asymptotically conformal curves having interior and exterior zero angles of the power type at the connecting points of boundary arcs.

Throughout this paper, $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (generally, different in different relations), which depend on G in general and on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

For any $k \geq 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k + 1, ..., m$. For any $i = 1, 2, ...,$ $k = 0, 1, 2$ and $\varepsilon_1 > 0$, we denote by $f_i : [0, \varepsilon_1] \to \mathbb{R}^+$ and $g_i : [0, \varepsilon_1] \to \mathbb{R}^+$ twice differentiable functions such that

$$f_i(0) = g_i(0) = 0,\ f_i^{(k)}(x) > 0,\ \ g_i^{(k)}(x) > 0,\ 0 < x \leq \varepsilon_1. \tag{5}$$

Definition 1 *We say that a Jordan region $G \in AC\,(f_i,\ g_i)$, for some $f_i = f_i(x)$, $i = \overline{1, m_1}$ and $g_i = g_i(x)$, $i = \overline{m_1 + 1, m}$, defined as in (5), if $L = \partial G = \bigcup_{i=0}^{m} L_i$ is the union of the finite number of asymptotically conformal arcs L_i, connecting at the points $\{z_i\}_{i=0}^m \in L$ and such that L is a locally asymptotically conformal arc at the $z_0 \in L\backslash\ \{z_i\}_{i=1}^m$ and, in the (x, y) local co-ordinate system with its origin at the z_i, $1 \leq i \leq m$, the following conditions are satisfied:*

a) for every $z_i \in L$, $i = \overline{1, m_1}$, $m_1 \leq m$,

$$\left\{z = x + iy : |z| \leq \varepsilon_1,\ \overset{i}{c}_{11} f_i(x) \leq y \leq \overset{i}{c}_{12} f_i(x),\ 0 \leq x \leq \varepsilon_1\right\} \subset \overline{G},$$
$$\{z = x + iy : |z| \leq \varepsilon_1,\ |y| \geq \varepsilon_2 x,\ 0 \leq x \leq \varepsilon_1\} \subset \overline{\Omega};$$

b) for every $z_i \in L$, $i = \overline{m_1 + 1, m}$,

$$\left\{z = x + iy : |z| < \varepsilon_3,\ \ c_{21}^i g_i(x) \leq y \leq c_{22}^i g_i(x),\ 0 \leq x \leq \varepsilon_3\right\} \subset \overline{\Omega},$$
$$\{z = x + iy : |z| < \varepsilon_3,\ \ |y| \geq \varepsilon_4 x,\ 0 \leq x \leq \varepsilon_3\} \subset \overline{G},$$

for some constants $-\infty < c_{11}^i < c_{12}^i < \infty$, $-\infty < c_{21}^i < c_{22}^i < \infty$ *and* $\varepsilon_s > 0$, $s = \overline{1,4}$.

Definition 2 *We say that a Jordan region* $G \in \widetilde{AC}\left(f_i, g_i\right)$, $f_i = f_i(x)$, $i = \overline{1, m_1}$, $g_i = g_i(x)$, $i = \overline{m_1 + 1, m}$, *if* $G \in AC\left(f_i, g_i\right)$ *and* $L := \partial G$ *is rectifiable.*

It is clear from Definitions 2 and 1, that each region $G \in \widetilde{AC}\left(f_i, g_i\right)$ may have m_1 interior and $m - m_1$ exterior zero angles (with respect to $\overline{G}$) at the points $\{z_i\}_{i=1}^m \in L$. If a region G does not have interior zero angles ($m_1 = 0$) (exterior zero angles ($m_1 = m$)), then it is written as $G \in \widetilde{AC}\left(0, g_i\right)$ ($G \in \widetilde{AC}\left(f_i, 0\right)$). If a domain G does not have such angles ($m = 0$), then we will assume that G is bounded by a rectifiable asymptotically conformal curves and in this case we set $\widetilde{AC}\left(0, 0\right) \equiv \widetilde{AC}$.

Throughout this work, we will assume that the points $\{\xi_i\}_{i=1}^m \in L$ defined in (1) and the points $\{z_i\}_{i=1}^m \in L$ defined in Definition 2 and 1 coincide. Without loss of generality, we also will assume that the points $\{z_i\}_{i=0}^m$ are ordered in the positive direction on the curve L such that G has interior zero angles at the points $\{z_i\}_{i=1}^{m_1}$, if $m_1 \geq 1$ and exterior zero angles at the points $\{z_i\}_{i=m_1+1}^m$, if $m \geq m_1 + 1$.

2 Main Results

Now, we can state our new results. Our first result (Nikolskii-type inequality) is related to the general case. Namely, let region G has $m_1 \geq 1$ interior zero angles at the points $\{z_i\}_{i=1}^{m_1}$ and $m - m_1$ exterior zero angles at the points $\{z_i\}_{i=m_1+1}^m$. In this case, we have the following estimate, i.e. with respect to each points $\{z_i\}_{i=1}^m$:

Theorem 1 *Let* $p > 0$; $G \in \widetilde{AC}\left(f_i, g_i\right)$, *for some* $f_i(x) = C_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = \overline{1, m_1}$, *and* $g_i(x) = C_i x^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{m_1 + 1, m}$; $h(z)$ *defined as in (1). Then, for any* $\gamma_i > -1$, $i = \overline{1, m}$, *and* $P_n \in \wp_n$, $n \in \mathbb{N}$, *there exists* $c_1 = c_1(G, p, \varepsilon, \gamma_i, \beta_i) > 0$ *such that the following*

$$\left\|P_n\right\|_\infty \leq c_1 \left(\sum_{i=1}^{m_1} n^{\frac{(\widetilde{\gamma}_i + 1)(1 + \widetilde{\varepsilon})}{p}} + \sum_{i=m_1+1}^{m} n^{\left(\frac{\widetilde{\gamma}_i}{1+\beta_i} + 1\right)\frac{1}{p} + \varepsilon} \right) \left\|P_n\right\|_p, \tag{6}$$

holds for $\widetilde{\varepsilon} := \begin{cases} \varepsilon, \text{ if } \alpha_1 = 0, \\ 1, \text{ if } \alpha_1 \neq 0, \end{cases}$ *and arbitrary small* $\varepsilon > 0$, *where* $\widetilde{\gamma}_i := \max\{0; \gamma_i\}$, $i = \overline{1, m}$.

Now, for simplicity of our presentations, we assume that: $i = 1, 2$; $m_1 = 1$, $m = 2$; i.e. our region G has one interior zero (or it does not exist) angle having " f_1−touching" with $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, at the point z_1 and exterior zero angle having "g_2−touching" with $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$, at the point z_2, for some constants $-\infty < C_i < +\infty$, $C_i := C_i(c_{i1}^i, c_{i2}^i)$, $i = 1, 2$, where the constants c_{ij}^i, $i, j = 1, 2$, are taken from Definition 2. In this case, combining the terms related to the interior and exterior zero angles, we obtain the following:

Theorem 2 *Let* $p > 0$; $G \in \widetilde{AC}\left(f_1, g_2\right)$, *for some* $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *defined as in (1) for* $m = 2$. *Then, for any* $\gamma_i > -1$, $i = 1, 2$, *and* $P_n \in \wp_n$, $n \in \mathbb{N}$, *there exists* $c_2 = c_2(G, p, \varepsilon, \gamma_i, \beta_2) > 0$ *such that:*

$$\left\|P_n\right\|_\infty \leq c_2 A_n \left\|P_n\right\|_p, \tag{7}$$

where

$$A_n := \begin{cases} n^{\frac{\gamma_1 + 2}{p}}, & \gamma_1 > \frac{\gamma_2}{1+\beta_2} - 1, \ \gamma_2 > 1 + \beta_2; \\ n^{\left(\frac{\gamma_2}{1+\beta_2} + 1\right)\frac{1}{p} + \varepsilon}, & 0 < \gamma_1 \leq \frac{\gamma_2}{1+\beta_2} - 1, \ \gamma_2 > 1 + \beta_2; \\ n^{\frac{\widetilde{\gamma}_1 + 2}{p}}, & \gamma_1 > -1, \ -1 < \gamma_2 < 1 + \beta_2. \end{cases} \tag{8}$$

In particular, if $\alpha_1 = 0$, i.e. G has only exterior zero angle at the z_2, then we have:

Theorem 3 *Let $p > 0$; $G \in \widetilde{AC}(0, g_2)$, for some $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ defined as in (1) for $m = 2$. Then, for any $\gamma_i > -1$, $i = 1, 2$, and $P_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_3 = c_3(G, p, \varepsilon, \gamma_i, \beta_2) > 0$ such that:*

$$\|P_n\|_\infty \leq c_3 A_n \|P_n\|_p ,$$

where

$$A_n = \begin{cases} n^{\frac{\gamma_1+1}{p}+\varepsilon}, & \gamma_1 > \frac{\gamma_2}{1+\beta_2},\ \gamma_2 > 0; \\ n^{\left(\frac{\gamma_2}{1+\beta_2}+1\right)\frac{1}{p}+\varepsilon}, & 0 < \gamma_1 \leq \frac{\gamma_2}{1+\beta_2},\ \gamma_2 > 0; \\ n^{\frac{1}{p}+\varepsilon}, & -1 < \gamma_1, \gamma_2 \leq 0, \end{cases} \tag{9}$$

The sharpness of the estimations (7)-(9) for some special cases can be discussed by comparing them with the following results:

Remark 1 *For the polynomials $P_n^*(z) = 1 + z + ... + z^n$, a) $h^*(z) \equiv 1$, b) $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$, and $L := \{z : |z| = 1\}$, there exists a constant $c_4 = c_4(p) > 0$ and $c_5 = c_5(h^{**}, p) > 0$ such that:*

$$a)\ \ \|P_n^*\|_{\mathcal{L}_\infty} \geq c_4 n^{\frac{1}{p}} \|P_n^*\|_{\mathcal{L}_p(1,\ L)}, \quad p > 1;$$
$$b)\ \ \|P_n^*\|_{\mathcal{L}_\infty} \geq c_5 n^{\frac{\gamma+1}{p}} \|P_n^*\|_{\mathcal{L}_p(h^{**},\ L)}, \quad p > \gamma + 1.$$

3 Some Aauxiliary Results

For $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

Let $G \subset \mathbb{C}$ be a bounded region, and $L := \partial G$ be a Jordan curve, $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$ ($\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Denote by $w = \Phi(z)$ the univalent conformal mapping of Ω onto $\Delta := \{w : |w| > 1\}$ with normalization $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$ and $\Psi := \Phi^{-1}$.

For $t \geq 1$, $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set:;

$$L_t\ :\ = \{z :\ |\Phi(z)| = t\}\ (L_1 \equiv L),\ G_t := intL_t,\ \Omega_t := extL_t;$$
$$d(z, M) = dist(z, M) := \inf\{|z - \zeta| :\ \zeta \in M\},$$

The following definitions of the K-quasiconformal curves are well known (see, for example, [7], [19, p.97] and [30]):

Definition 3 *The Jordan arc (or curve) L is called $K-$quasiconformal ($K \geq 1$), if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).*

Let $F(L)$ denote the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and let

$$K_L := \inf\{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a $K-$quasiconformal curve, if $K_L \leq K$.

Lemma 1 *[1] Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon_1} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^c,$$

where $\varepsilon_1 < 1$, $c > 1$, $0 < r_0 < 1$ are constants, depending on G and $L_{r_0} := \{z = \psi(w) : |w| = r_0\}$.

Lemma 2 *[20, p.342] Let L be an asymptotically conformal curve. Then, Φ and Ψ are Lipα for all $\alpha < 1$ in $\overline{\Omega}$ and $\overline{\Delta}$, correspondingly.*

Lemma 3 *Let L be an asymptotically conformal curve. Then,*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{1+\varepsilon},$$

for all $w_1, w_2 \in \overline{\Delta}$ and $\forall\varepsilon > 0$.

This fact follows from Lemma 2. We also will use the estimation for the Ψ' (see, for example, [9, Th.2.8]):

$$|\Psi'(\tau)| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{10}$$

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of the points on L and the weight function $h(z)$ defined as (1).

Lemma 4 *Let L be a rectifiable Jordan curve; $h(z)$ defined as in (1). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$, we have*

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \le R^{n+\frac{1+\widetilde{\gamma}}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \ p > 0, \tag{11}$$

where $\widetilde{\gamma} := \max\left\{0; \gamma_i : \ i = \overline{1,m}\right\}$.

Remark 2 *In case of $h(z) \equiv 1$, the estimate (11) has been proved in [17].*

4 Proof of Theorems

4.1 Proof of Theorems 1-3.

Proof. Let $G \in \widetilde{AC}(f_i, g_i)$, for some $f_i(x) = c_i x^{1+\alpha_i}$, $\alpha_i \ge 0$, $i = \overline{1, m_1}$, and $g_i(x) = c_i x^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{m_1+1, m}$, be given. Let $w = \varphi_R(z)$ be the univalent conformal mapping of G_R, $R > 1$, onto the B normalized by $\varphi_R(0) = 0$, $\varphi_R'(0) > 0$, and let $\{\zeta_j\}$, $1 \le j \le m \le n$, be zeros of $P_n(z)$ lying on G_R. Let

$$B_{m,R}(z) := \prod_{j=1}^{m} b_{j,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \overline{\varphi_R(\zeta_j)}\varphi_R(z)}$$

denote a Blaschke function with respect to zeros $\{\zeta_j\}$, $1 \le j \le m \le n$, of $P_n(z)$.

Let us set:

$$Q_n(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}, \ p > 0, \ z \in G_R.$$

The function $Q_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. Then, Cauchy integral representation for the $Q_n(z)$ in G_R gives:

$$Q_n(z) = \frac{1}{2\pi i}\int_{L_R} Q_n(\zeta)\frac{d\zeta}{\zeta - z}, \ z \in G_R,$$

or

$$\left|\left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}\right| \le \frac{1}{2\pi}\int_{L_R}\left|\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta - z|} \le \int_{L_R}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta - z|},$$

since $|B_{m,R}(\zeta)| = 1$, for $\zeta \in L_R$. Let now $z \in L$. Multiplying the numerator and determinator of the integrand by $h^{1/2}(\zeta)$, according to the Hölder inequality, we obtain:

$$\left|\frac{P_n(z)}{B_{m,R}(z)}\right|^{p/2} \le \frac{1}{2\pi}\left(\int_{L_R} h(\zeta)|P_n(\zeta)|^p|d\zeta|\right)^{1/2} \tag{12}$$

$$\times\left(\int\limits_{L_R}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}|\zeta-z_j|^{\gamma_j}\,|\zeta-z|^2}\right)^{1/2}=:\frac{1}{2\pi}J_{n,1}\times J_{n,2},$$

where

$$J_{n,1}:=\left(\int\limits_{L_R}h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta|\right)^{1/2},\ J_{n,2}:=\left(\int\limits_{L_R}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}|\zeta-z_j|^{\gamma_j}\,|\zeta-z|^2}\right)^{1/2}.$$

Then, since $|B_{m,R}(z)|<1$, for $z\in L$, from Lemma 4, we have:

$$|P_n(z)|\preceq(J_{n,1}\cdot J_{n,2})^{2/p}\preceq\|P_n\|_p\cdot J_{n,2}^{2/p},\ z\in L. \tag{13}$$

To estimate the integral $J_{n,2}$, we introduce:

$$w_j:=\Phi(z_j),\ \varphi_j:=\arg w_j,\ L_R^j:=L_R\cap\overline{\Omega}^j,\ j=\overline{1,m},$$

where $\Omega^j:=\Psi(\Delta_j')$;

$$\Delta_1':=\left\{t=Re^{i\theta}:R>1,\ \frac{\varphi_m+\varphi_1}{2}\le\theta<\frac{\varphi_1+\varphi_2}{2}\right\},$$
$$\Delta_m':=\left\{t=Re^{i\theta}:R>1,\ \frac{\varphi_{m-1}+\varphi_m}{2}\le\theta<\frac{\varphi_m+\varphi_1}{2}\right\}.$$

and, for $j=\overline{2,m-1}$

$$\Delta_j':=\left\{t=Re^{i\theta}:R>1,\ \frac{\varphi_{j-1}+\varphi_j}{2}\le\theta<\frac{\varphi_j+\varphi_{j+1}}{2}\right\}.$$

Then, we get

$$J_{n,2}^2=\sum_{i=1}^{m}\int\limits_{L_R^i}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}|\zeta-z_j|^{\gamma_j}\,|\zeta-z|^2}\asymp\sum_{i=1}^{m}\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta-z_i|^{\gamma_i}\,|\zeta-z|^2}=:\sum_{i=1}^{m}J_{n,2}^i, \tag{14}$$

where

$$J_{n,2}^i:=\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta-z_i|^{\gamma_i}\,|\zeta-z|^2},\ i=\overline{1,m}, \tag{15}$$

since the points $\{z_j\}_{j=1}^m\in L$ are distinct. It remains to estimate the integrals $J_{n,2}^i$ for each $i=\overline{1,m}$. For simplicity of our next calculations, we assume that:

$$i=1,2;\ m_1=1,\ m=2;\ \ z_1=-1,\ z_2=1;\ (-1,1)\subset G;\ R=1+\frac{\varepsilon_0}{n}, \tag{16}$$

and let local co-ordinate axis in Definitions 1 and 2 is parallel to OX and OY in the OXY co-ordinate system; $L=L^+\cup L^-$, where $L^+:=\{z\in L:\mathrm{Im}z\ge0\}$, $L^-:=\{z\in L:\mathrm{Im}z<0\}$. Let $w^\pm:=\left\{w=e^{i\theta}:\theta=\frac{\varphi_1\pm\varphi_2}{2}\right\}$, $z^\pm\in\Psi(w^\pm)$ and L^i an arcs, connecting the points $z^+,\ z_i,\ z^-\in\ L$; $L^{i,\pm}:=L^i\cap L^\pm,\ i=1,2$. Let z_0 be taken as an arbitrary point on L^+ (or on L^- subject to the chosen direction). For simplicity, without loss of generality, we assume that $z_0=z^+\ \ (z_0=z^-\ )$.

Analogously to the previous notations, we introduce the following: $L_R=L_R^+\cup L_R^-$, where $L_R^+:=\{z\in L_R:\mathrm{Im}z\ge0\}$, $L_R^-:=\{z\in L_R:\mathrm{Im}z<0\}$; Let $w_R^\pm:=\left\{w=Re^{i\theta}:\theta=\frac{\varphi_1\pm\varphi_2}{2}\right\}$, $z_R^\pm\in\Psi(w_R^\pm)$.We set: $z_{i,R}\in L_R$, such that $d_{i,R}=|z_i-z_{i,R}|$ and $\zeta^\pm\in L^\pm$, such that $d(z_{2,R},L^2\cap L^\pm):=d(z_{2,R},L^\pm)$; $z_i^\pm:=\left\{\zeta\in L^i:|\zeta-z_i|=c_id(z_i,L_R)\right\}$, $z_{i,R}^\pm:=\left\{\zeta\in L_R^i:\ |\zeta-z_{i,R}|=c_id(z_{i,R},L_R)\right\}$, $w_{i,R}^\pm=\Phi(z_{i,R}^\pm)$. Let

$L_R^i,\ i=1,2$, denote arcs, connecting the points $z_R^+,\ z_{i,R},\ z_R^- \in\ L_R,\ L_R^{i,\pm} := L_R^i \cap L_R^\pm$ and $l_{i,R}^\pm(z_{i,R}^\pm, z_R^\pm)$ denote arcs, connecting the points $z_{i,R}^\pm$ with $z_R^\pm$, respectively and $\left| l_{i,R}^\pm \right| := mes\ l_{i,R}^\pm(z_{i,R}^\pm, z_R^\pm),\ i=1,2.$ We denote:

$$\begin{aligned}
S_{1,R}^{i,\pm} &:= \left\{\zeta \in L_R^{i,\pm}:\ |\zeta - z_i| < c_i d_{i,R}\right\},\\
S_{2,R}^{i,\pm} &:= \left\{\zeta \in L_R^{i,\pm}: c_i d_{i,R} \le |\zeta - z_i| \le \left| l_{i,R}^\pm \right|\right\},\ \mathcal{F}_{j,R}^{i,\pm} := \Phi(S_{j,R}^{i,\pm});\\
S_1^{i,\pm} &:= \left\{\zeta \in L^{i,\pm}:\ |\zeta - z_i| < c_i d_{i,R}\right\},\\
S_2^{i,\pm} &:= \left\{\zeta \in L^{i,\pm}: c_i d_{i,R} \le |\zeta - z_i| \le \left| l_{i,R}^\pm \right|\right\},\ \mathcal{F}_j^{i,\pm} := \Phi(S_j^{i,\pm}),\ i,j=1,2.
\end{aligned}$$

Taking into consideration these designations and replacing the variable $\tau = \Phi(\zeta)$, from (10) and (15), we have:

$$\begin{aligned}
J_{n,2}^i &\asymp \sum_{i,j=1}^{2} \int\limits_{\mathcal{F}_{j,R}^{i,+} \cup \mathcal{F}_{j,R}^{i,-}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{\gamma_i}\,|\Psi(\tau)-\Psi(w')|^2} \qquad (17)\\
&\asymp \sum_{i,j=1}^{2} \int\limits_{\mathcal{F}_{j,R}^{i,+} \cup \mathcal{F}_{j,R}^{i,-}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{\gamma_i}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)}\\
&=: \sum_{i,j=1}^{2} \left[J(\mathcal{F}_{j,R}^{i,+}) + J(\mathcal{F}_{j,R}^{i,-})\right].
\end{aligned}$$

So, we need to evaluate the integrals $J(\mathcal{F}_{j,R}^{i,+})$ and $J(\mathcal{F}_{j,R}^{i,-})$ for each $i,j=1,2$. For this, we will continue in the following manner. Let

$$\|P_n\|_\infty =: |P_n(z')|,\ z' \in L, \qquad (18)$$

and let $w' = \Phi(z')$. There are two possible cases: the point z' may lie on L^1 or L^2.

1) Suppose first that $z' \in L^1$. If $z' \in S_i^{1,\pm}$, then $w' \in \mathcal{F}_i^{1,\pm}$, for $i=1,2$. Consider the individual cases.

1.1) If $z' \in S_1^{1,\pm}$, then $w' \in \mathcal{F}_1^{1,\pm}$ and

$$J(\mathcal{F}_{1,R}^{1,+}) + J(\mathcal{F}_{1,R}^{1,-}) \qquad (19)$$

$$\begin{aligned}
&\preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{\left[\min\left\{|\Psi(\tau)-\Psi(w_1)|\,;|\Psi(\tau)-\Psi(w')|\right\}\right]^{\gamma_1+1}}\\
&\preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{\left[\min\left\{|\tau-w_1|\,;|\tau-w'|\right\}\right]^{(\gamma_1+1)(1+\widetilde{\varepsilon})}} \preceq n^{(\gamma_1+1)(1+\widetilde{\varepsilon})},
\end{aligned}$$

for $\gamma_1 > 0$, and

$$J(\mathcal{F}_{1,R}^{1,+}) + J(\mathcal{F}_{1,R}^{1,-}) \preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)}\,|d\tau|}{|\Psi(\tau)-\Psi(w')|} \qquad (20)$$

$$\preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|} \preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{|\tau-w'|^{1+\widetilde{\varepsilon}}} \preceq n^{1+\widetilde{\varepsilon}},$$

for $-1 < \gamma_1 \le 0$;

1.2) If $z' \in S_2^{1,\pm}$, then

$$J(\mathcal{F}_{1,R}^{1,+}) + J(\mathcal{F}_{1,R}^{1,-}) \preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|} \qquad (21)$$

$$\preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+}\cup\mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{[\min\{|\tau-w_1|\,;|\tau-w'|\}]^{(\gamma_1+1)(1+\widetilde{\varepsilon})}} \preceq n^{(\gamma_1+1)(1+\widetilde{\varepsilon})},$$

for all $\gamma_1 > 0$ and

$$J(\mathcal{F}_{1,R}^{1,+}) + J(\mathcal{F}_{1,R}^{1,-}) \preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+}\cup\mathcal{F}_{1,R}^{1,-}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)}\,|d\tau|}{|\Psi(\tau)-\Psi(w')|} \tag{22}$$

$$\preceq n \int\limits_{\mathcal{F}_{1,R}^{1,+}\cup\mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{|\tau-w'|^{1+\widetilde{\varepsilon}}} \preceq n^{1+\widetilde{\varepsilon}},$$

for $-1 < \gamma_1 \leq 0$;

1.3) If $z' \in S_1^{1,\pm}$,then

$$J(\mathcal{F}_{2,R}^{1,+}) + J(\mathcal{F}_{2,R}^{1,-}) \preceq n \int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|} \tag{23}$$

$$\preceq n \int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}} \frac{|d\tau|}{\min\{|\tau-w_1|\,;|\tau-w'|\}^{(\gamma_1+1)(1+\widetilde{\varepsilon})}} \preceq n^{(\gamma_1+1)(1+\widetilde{\varepsilon})},$$

for $\gamma_1 > 0$ and

$$J(\mathcal{F}_{2,R}^{1,+}) + J(\mathcal{F}_{2,R}^{1,-}) \preceq n \int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)}\,|d\tau|}{|\Psi(\tau)-\Psi(w')|} \tag{24}$$

$$\preceq n \int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}} \frac{|d\tau|}{|\tau-w'|^{1+\widetilde{\varepsilon}}} \preceq n^{1+\widetilde{\varepsilon}},$$

for $-1 < \gamma_1 \leq 0$;

1.4) If $z' \in S_2^{1,\pm}$, then

$$J(\mathcal{F}_{2,R}^{1,+}) + J(\mathcal{F}_{2,R}^{1,-}) \preceq n \int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|} \tag{25}$$

$$\preceq n \int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}} \frac{|d\tau|}{[\min\{|\tau-w_1|\,;|\tau-w'|\}]^{[\gamma_1+1](1+\widetilde{\varepsilon})}} \preceq n^{(\gamma_1+1)(1+\widetilde{\varepsilon})},$$

for $\gamma_1 > 0$, and

$$J(\mathcal{F}_{2,R}^{1,+}) + J(\mathcal{F}_{2,R}^{1,-}) \preceq n \int\limits_{\mathcal{F}_{2,R}^{1,+}\cup\mathcal{F}_{2,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|} \preceq n^{1+\widetilde{\varepsilon}}, \tag{26}$$

for $-1 < \gamma_1 \leq 0$. Combining the relations (19)-(26), we obtain:

$$\sum_{i=1}^{2} \left[J(\mathcal{F}_{i,R}^{1,+}) + J(\mathcal{F}_{i,R}^{1,-})\right] \preceq n^{(\gamma_1+1)(1+\widetilde{\varepsilon})}, \tag{27}$$

for $\gamma_1 > 0$ and

$$\sum_{i=1}^{2} \left[J(\mathcal{F}_{i,R}^{1,+}) + J(\mathcal{F}_{i,R}^{1,-})\right] \preceq n^{1+\widetilde{\varepsilon}}, \tag{28}$$

for $-1 < \gamma_1 \le 0$.

Therefore, in case of $z' \in L^1$ for each $\gamma_1 > -1$, from (17), (27) and (28) we get:

$$J^1_{n,2} \preceq n^{(\widetilde{\gamma_1+1})(1+\widetilde{\varepsilon})}. \tag{29}$$

2) Now, suppose that $z' \in L^2$. If $z' \in S_i^{2,\pm}$, then $w' \in \mathcal{F}_i^{2,\pm}$, for $i = 1, 2$. For the estimate of $J^i_{n,2}$ from (17), again we will consider individual cases.

2.1) If $z' \in S_1^{2,\pm}$, then

$$J(\mathcal{F}^{2,+}_{1,R}) + J(\mathcal{F}^{2,-}_{1,R}) = \tag{30}$$

$$\preceq n \int\limits_{\mathcal{F}^{2,+}_{1,R}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2} |\Psi(\tau) - \Psi(w')|} + n \int\limits_{\mathcal{F}^{2,-}_{1,R}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2} |\Psi(\tau) - \Psi(w')|},$$

for all $\gamma_2 > -1$. The last two integrals are evaluated identically. Therefore, we evaluate one of them, say the first. When $\tau \in \mathcal{F}^{2,+}_{1,R}$, for the $|\Psi(\tau) - \Psi(w')|$, we obtain:

$$\begin{aligned} |\Psi(\tau) - \Psi(w')| &\succeq \max\left\{ |\Psi(\tau) - \Psi(w_2)| ; \ \left|\Psi(\tau) - z_2^+\right| \right\} \\ &= |\Psi(\tau) - \Psi(w_2)| \succeq \left|\Psi(\tau) - z_2^+\right|^{\frac{1}{1+\beta_2}}. \end{aligned}$$

Then,

$$\begin{aligned} J(\mathcal{F}^{2,+}_{1,R}) &\preceq n \int\limits_{\mathcal{F}^{2,+}_{1,R}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{\frac{\gamma_2+1}{1+\beta_2}}} \preceq n \int\limits_{\mathcal{F}^{2,+}_{1,R}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{\gamma_2+1}{1+\beta_2}+\varepsilon}} \\ &\preceq \begin{cases} n^{\frac{\gamma_2+1}{1+\beta_2}+\varepsilon}, & \frac{\gamma_2+1}{1+\beta_2} > 1-\varepsilon, \\ n\ln n, & \frac{\gamma_2+1}{1+\beta_2} = 1-\varepsilon, \\ n, & \frac{\gamma_2+1}{1+\beta_2} < 1-\varepsilon, \end{cases} \end{aligned}$$

if $\gamma_2 > 0$, and

$$J(\mathcal{F}^{2,+}_{1,R}) \preceq n \int\limits_{\mathcal{F}^{2,+}_{1,R}} \frac{|\Psi(\tau) - \Psi(w_2)|^{(-\gamma_2)} |d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{\frac{1}{1+\beta_2}}} \preceq n \int\limits_{\mathcal{F}^{2,+}_{1,R}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{1+\varepsilon}{1+\beta_2}}} \preceq n^{\frac{1+\varepsilon}{1+\beta_2}},$$

if $-1 < \gamma_2 \le 0$, and so, in this case, we get:

$$J(\mathcal{F}^{2,+}_{1,R}) + J(\mathcal{F}^{2,-}_{1,R}) \preceq \begin{cases} n^{\frac{\gamma_2+1}{1+\beta_2}+\varepsilon}, & \frac{\gamma_2+1}{1+\beta_2} > 1-\varepsilon, \\ n\ln n, & \frac{\gamma_2+1}{1+\beta_2} = 1-\varepsilon, \\ n, & \frac{\gamma_2+1}{1+\beta_2} < 1-\varepsilon, \end{cases} \tag{31}$$

if $\gamma_2 > 0$, and

$$J(\mathcal{F}^{2,+}_{1,R}) + J(\mathcal{F}^{2,-}_{1,R}) \preceq n^{\frac{1+\varepsilon}{1+\beta_2}},$$

if $-1 < \gamma_2 \le 0$.

2.2) If $z' \in S_2^{2,\pm}$, then

$$J(\mathcal{F}^{2,+}_{1,R}) + J(\mathcal{F}^{2,-}_{1,R}) \preceq n \int\limits_{\mathcal{F}^{2,+}_{1,R} \cup \mathcal{F}^{2,-}_{1,R}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2} |\Psi(\tau) - \Psi(w')|},$$

for all $\gamma_2 > -1$. When $\tau \in \mathcal{F}_{1,R}^{2,+}$ for the $|\Psi(\tau) - \Psi(w')|$, we obtain:

$$|\Psi(\tau) - \Psi(w')| \succeq \left|\Psi(\tau) - z_2^+\right|$$

and, analogous to previous case, we get:

$$J(\mathcal{F}_{1,R}^{2,+}) \preceq n \int\limits_{\mathcal{F}_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2} \left|\Psi(\tau) - z_2^+\right|} \preceq n \int\limits_{\mathcal{F}_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{\gamma_2}{1+\beta_2}+1+\varepsilon}} \preceq n^{\frac{\gamma_2}{1+\beta_2}+1+\varepsilon},$$

if $\gamma_2 > 0$, and

$$J(\mathcal{F}_{1,R}^{2,+}) \preceq n \int\limits_{\mathcal{F}_{1,R}^{2,+}} \frac{|\Psi(\tau) - \Psi(w_2)|^{(-\gamma_2)} |d\tau|}{\left|\Psi(\tau) - z_2^+\right|} \preceq n \int\limits_{\mathcal{F}_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|} \preceq n^{1+\varepsilon},$$

if $-1 < \gamma_2 \leq 0$. So, in this case we have:

$$J(\mathcal{F}_{1,R}^{2,+}) + J(\mathcal{F}_{1,R}^{2,-}) \preceq n^{\frac{\gamma_2}{1+\beta_2}+1+\varepsilon}, \tag{32}$$

if $\gamma_2 > 0$, and

$$J(\mathcal{F}_{1,R}^{2,+}) + J(\mathcal{F}_{1,R}^{2,-}) \preceq n^{1+\varepsilon},$$

if $-1 < \gamma_2 \leq 0$.

2.3) If $z' \in S_1^{2,\pm}$, then

$$J(\mathcal{F}_{2,R}^{2,+}) + J(\mathcal{F}_{2,R}^{2,-}) \preceq n \int\limits_{\mathcal{F}_{2,R}^{2,+} \cup \mathcal{F}_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2} |\Psi(\tau) - \Psi(w')|} \tag{33}$$

$$\preceq n \int\limits_{\mathcal{F}_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2} |\Psi(\tau) - \Psi(w')|} + n \int\limits_{\mathcal{F}_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2} |\Psi(\tau) - \Psi(w')|},$$

for $\gamma_2 > 0$. The last two integrals are evaluated identically. Let's estimate first integral. For $\tau \in \mathcal{F}_{2,R}^{2,+}$ and $z' \in S_1^{2,\pm}$, we have:

$$|\Psi(\tau) - \Psi(w')| \succeq \left|\Psi(\tau) - z_2^+\right|;$$
$$|\Psi(\tau) - \Psi(w_2)| \succeq d_{2,R} \succeq \left|z_{2,R} - z_2^+\right|^{\frac{1}{1+\beta_2}} \succeq \left(\frac{1}{n}\right)^{\frac{1+\varepsilon}{1+\beta_2}}.$$

Then,

$$J(\mathcal{F}_{2,R}^{2,+}) \preceq n \int\limits_{\mathcal{F}_{2,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{\gamma_2} \left|\Psi(\tau) - z_2^+\right|} \preceq n^{\frac{\gamma_2}{1+\beta_2}+1+\varepsilon} \int\limits_{\mathcal{F}_{2,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{1+\varepsilon}} \preceq n^{\frac{\gamma_2}{1+\beta_2}+1+\varepsilon},$$

and so, for $\gamma_2 > 0$, we obtain:

$$J(\mathcal{F}_{2,R}^{2,+}) + J(\mathcal{F}_{2,R}^{2,-}) \preceq n^{\frac{\gamma_2}{1+\beta_2}+1+\varepsilon},$$

For $-1 < \gamma_2 \leq 0$, we get:

$$J(\mathcal{F}_{2,R}^{2,+}) + J(\mathcal{F}_{2,R}^{2,-}) = \int\limits_{\mathcal{F}_{2,R}^{2,+} \cup \mathcal{F}_{2,R}^{2,-}} \frac{|\Psi(\tau) - \Psi(w_2)|^{(-\gamma_2)} |\Psi'(\tau)| |d\tau|}{|\Psi(\tau) - \Psi(w')|^2} \tag{34}$$

$$\preceq n \int\limits_{\mathcal{F}_{2,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|} \preceq n \int\limits_{\mathcal{F}_{2,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{1+\varepsilon}} \preceq n^{1+\varepsilon},$$

Then, in this case, we have:

$$J(\mathcal{F}_{2,R}^{2,+}) + J(\mathcal{F}_{2,R}^{2,-}) \preceq n^{\frac{\gamma_2}{1+\beta_2}+1+\varepsilon}. \tag{35}$$

2.4) If $z' \in S_2^{2,+}$, then for $\gamma_2 > 0$

$$J(\mathcal{F}_{2,R}^{2,+}) \preceq \frac{n}{d_{2,R}^{\gamma_2}} \int\limits_{\mathcal{F}_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|} \tag{36}$$

$$\preceq n^{1+\frac{\gamma_2}{1+\beta_2}(1+\varepsilon)} \int\limits_{\mathcal{F}_{2,R}^{2,+}} \frac{|d\tau|}{|\tau - w'|^{1+\varepsilon}} \preceq n^{\frac{\gamma_2}{1+\beta_2}+1+\varepsilon},$$

and

$$J(\mathcal{F}_{2,R}^{2,-}) \preceq \frac{n}{d_{2,R}^{\gamma_2}} \int\limits_{\mathcal{F}_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|} \tag{37}$$

$$\preceq n^{1+\frac{\gamma_2}{1+\beta_2}+\varepsilon} \int\limits_{\mathcal{F}_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|} \preceq n^{1+\frac{\gamma_2}{1+\beta_2}+\varepsilon} \int\limits_{\mathcal{F}_{2,R}^{2,-}} \frac{|d\tau|}{|\tau - w'|^{1+\varepsilon}}$$

$$\preceq n^{\frac{\gamma_2}{1+\beta_2}+1+\varepsilon}.$$

Case of $z' \in S_2^{2,-}$ is absolutely identical to the case $z' \in S_2^{2,+}$. If $-1 < \gamma_2 \leq 0$, then

$$J(\mathcal{F}_{2,R}^{2,+}) = \int\limits_{\mathcal{F}_{2,R}^{2,+}} \frac{|\Psi(\tau) - \Psi(w_2)|^{(-\gamma_2)} |\Psi'(\tau)| \, |d\tau|}{|\Psi(\tau) - \Psi(w')|^2} \tag{38}$$

$$\preceq n \int\limits_{\mathcal{F}_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|} \preceq n^{1+\varepsilon},$$

and

$$J(\mathcal{F}_{2,R}^{2,-}) = \int\limits_{\mathcal{F}_{2,R}^{2,-}} \frac{|\Psi(\tau) - \Psi(w_2)|^{(-\gamma_2)} |\Psi'(\tau)| \, |d\tau|}{|\Psi(\tau) - \Psi(w')|^2} \tag{39}$$

$$\preceq n \int\limits_{\mathcal{F}_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|} \preceq n^{1+\varepsilon}.$$

Combining the estimations (17), (31)-(39), we obtain:

$$J_{n,2}^2 \preceq n^{1+\varepsilon},$$

for each $-1 < \gamma_2 \leq 0$ and

$$J_{n,2}^2 \preceq n^{\frac{\gamma_2}{(1+\beta_2}+1+\varepsilon}, \tag{40}$$

for each $\gamma_2 > 0$. Combining (40) and (29), for $m_1 = 1$, $m_2 = 1$, and any $p > 0$, we get:

$$J_{n,2}^1 + J_{n,2}^2 \preceq n^{1+\widetilde{\varepsilon}} + n^{1+\varepsilon}, \tag{41}$$

for each $-1 < \gamma_1 \leq 0$, $-1 < \gamma_2 \leq 0$ and

$$J_{n,2}^1 + J_{n,2}^2 \tag{42}$$

$$\preceq n^{\gamma_1+1+\widetilde{\varepsilon}} + n^{\frac{\gamma_2}{1+\beta_2}+1+\varepsilon},$$

for each $\gamma_1 > 0$, $\gamma_2 > 0$, where $\widetilde{\varepsilon} := \begin{cases} \varepsilon, \text{ if } \alpha_1 = 0, \\ 1, \text{ if } \alpha_1 \neq 0, \end{cases}$ and $p > 0$. Then, from (12)-(17), (41) and (42), for all $z \in L$, we obtain:

$$|P_n(z)| \preceq \|P_n\|_p \cdot \left(n^{\frac{\widetilde{\gamma_1+1+\widetilde{\varepsilon}}}{p}} + n^{\left(\widetilde{\frac{\gamma_2}{1+\beta_2}}+1\right)\frac{1}{p}+\varepsilon} \right)$$

$$\preceq \|P_n\|_p \cdot \begin{cases} n^{\frac{\gamma_1+2}{p}}, & \gamma_1 > \frac{\gamma_2}{1+\beta_2} - 1,\ \gamma_2 > 1+\beta_2; \\ n^{\frac{\gamma_1+2}{p}}, & \gamma_1 > 0,\ 0 < \gamma_2 < 1+\beta_2; \\ n^{\left(\frac{\gamma_2}{1+\beta_2}+1\right)\frac{1}{p}+\varepsilon}, & 0 < \gamma_1 \leq \frac{\gamma_2}{1+\beta_2} - 1,\ \gamma_2 > 1+\beta_2; \\ n^{\frac{2}{p}}, & -1 < \gamma_1 \leq 0,\ -1 < \gamma_2 < 1+\beta_2; \end{cases}$$

if $\alpha_1 \neq 0$, and

$$|P_n(z)| \preceq \|P_n\|_p \cdot \begin{cases} n^{\frac{\gamma_1+1}{p}+\varepsilon}, & \gamma_1 > \frac{\gamma_2}{1+\beta_2},\ \gamma_2 > 0; \\ n^{\left(\frac{\gamma_2}{1+\beta_2}+1\right)\frac{1}{p}+\varepsilon}, & 0 < \gamma_1 \leq \frac{\gamma_2}{1+\beta_2},\ \gamma_2 > 0; \\ n^{\frac{1}{p}+\varepsilon}, & -1 < \gamma_1, \gamma_2 \leq 0, \end{cases}$$

if $\alpha_1 = 0$. Therefore, we completed the proof.

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