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The Best Linear Approximation Methods of Analytic Functions in the Disk and Exact Values of Widths of Some Classes Functions in Hardy Spaces

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Abstract The best linear of approximation methods classes of analytic in the unit circle are constructed, and their averaged values of the moduli of continuity rth derivatives are majorised by a given function. The obtained results make it possible to calculate the exact values of different n-widths of classes of functions on the mentioned classes.

Keywords: Best linear approximation methods, classes of analytic functions, Hardy space, polynomial, modulus of smoothness, inequality, widths.

1 Introduction

Extreme problem of finding the best linear approximation methods for classes of analytic functions presents certain interest in the calculation of Gel'fand and linear n-widths. In this direction, the study has a number of final results (see., eg, [1,2,?,4,6,5,7,8,?,10,11,12,13,14] and references cited therein).

In this paper we construct the best linear approximation methods for certain classes of analytic functions, previously studied in [15,10], and calculated the exact value of a number of n-widths of classes of functions stated in more general Hardy spaces $H_{q,\rho}$ $(1 \le q \le \infty,\ 0 < \rho \le 1)$.

It is said that the analytic in the unit disk $|z| < 1$ the function f belongs to the Banach space H_q, if

$$\|f\|_q = \|f\|_{H_q} := \lim_{\rho \to 1-0} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{it})|^q dt \right)^{1/q} < \infty, \quad 1 \le q \le \infty.$$

The norm of the function $f \in H_q$, $1 \le q \le \infty$ is realized on its angular boundary values of $F(t) := f(e^{it})$. In the case of $q = \infty$ we assume additionally the function of f is continuous in the closed circle $|z| \le 1$. Let $H_{q,\rho}$ $(1 \le q \le \infty,\ 0 < \rho \le 1)$ be Hardy space of analytic in the disk $|z| < \rho$ functions f, for which $\|f\|_{H_{q,\rho}} = \|f(\rho\cdot)\|_{H_q} < \infty$.

Let $\mathscr{P}_n$ be a set of algebraic polynomials of degree at most n. By symbol

$$E_n(f)_q = \inf \left\{ \|f - p_{n-1}\|_q : \ p_{n-1} \in \mathscr{P}_{n-1} \right\}$$

we denote the best approximation of the function $f \in H_q$ by setting $\mathscr{P}_{n-1}$ of polynomials degree $\le n-1$. The derivative rth order of the function f in the argument z is denoted as usual $f^{(r)} := d^r f/dz^r$ $(r \in \mathbb{N},\ f^{(0)} = f)$.

Structural properties of the function $f^{(r)} \in H_q$ are characterized by rate which tends to zero modulus of smoothness of its boundary values of the derivative

$$\omega_2(F^{(r)}; 2t)_q := \sup \Big\{ \|F^{(r)}(\cdot + \tau) - 2F^{(r)}(\cdot) + F^{(r)}(\cdot - \tau)\|_q : \ |\tau| \le t \Big\},$$

as $t \to 0$, setting this rate of decay to zero through a majorant averaged values containing $\omega_2(F^{(r)}; 2t)_{H_q}$.

Let $\Phi(x)$ $(x \geq 0)$ be arbitrary positive nondecreasing function such as $\Phi(0) = 0$. For any given value of the parameter $\mu \geq 1/2$, through $W_q^{(r)}(\Phi; \mu)$ $(r \in \mathbb{N},\ 1 \leq q \leq \infty)$ denote the class of functions $f \in H_q$, for which the derivative $f^{(r)} \in H_q$ for any $h \in (0, \pi]$ satisfies the condition

$$\frac{1}{h}\int_0^h \omega_2(F^{(r)}; 2t)_q \left(1 + (\mu^2 - 1)\sin\frac{\pi t}{2h}\right) dt \leq \Phi(h).$$

Here further definition and notation are necessary. Let X be Banach space; S the unit ball in it; $\mathfrak{M}$ convex centrally symmetric subset of X; $L_n \subset X$ n-dimensional subspace; $V(f, L_n)$ a continuous linear operator that maps X to L_n; L^n linear subspace of codimension n from X. Let

$$E(f, L_n)_X = \inf\Big\{\|f - \varphi\|_X :\ \varphi \in L_n\Big\}$$

be the best approximation of a function $f \in X$, and through

$$\mathscr{E}_n(f, V(f))_X := \mathscr{E}(f, V(f, L_n))_X = \|f - V(f, L_n)\|_X$$

deviation of $f \in X$ of a continuous linear operator is denoted as $V(f, L_n)$ in X. For introduced the above set of $\mathfrak{M} \subset X$ we get

$$E(\mathfrak{M}, L_n)_X \overset{def}{=} \sup\Big\{\ E(f, L_n)_X :\ f \in \mathfrak{M}\Big\},$$

$$\mathscr{E}(\mathfrak{M}, V, L_n)_X \overset{def}{=} \sup\Big\{\ \mathscr{E}(f, V(f, L_n))_X :\ f \in \mathfrak{M}\Big\}.$$

The magnitudes

$$b_n(\mathfrak{M}; X) = \sup\{\sup\{\varepsilon > 0 :\ \varepsilon S \cap L_{n+1} \subset \mathfrak{M}\} :\ L_{n+1} \subset X\},$$

$$d_n(\mathfrak{M}; X) = \inf\{\ E_n(\mathfrak{M}, L_n)_X :\ L_n \subset X\},$$
$$d^n(\mathfrak{M}; X) = \inf\{\sup\{\|f\|_X :\ f \in \mathfrak{M} \cap L^n\} :\ L^n \subset X\},$$
$$\delta_n(\mathfrak{M}; X) = \inf\{\inf\{\mathscr{E}_n(\mathfrak{M}, V, L_n)_X :\ V : X \to L_n\} :\ L_n \subset X\}$$

are respectively called Bernstein, Kolmogorov, Gel'fand and linear n-widths. Between the above n-widths for any centrally symmetric compact set $\mathfrak{M} \subset X$ are true the relations (see., eg, [4,16]):

$$b_n(\mathfrak{M}; X) \leq \begin{matrix} d_n(\mathfrak{M}; X) \\ d^n(\mathfrak{M}; X) \end{matrix} \leq \delta_n(\mathfrak{M}; X). \tag{1}$$

From the results [15, p. 93] and Corollary 3 [17, p. 289] follows that if for a given $\mu \geq 1/2$, any $\tau \in (0, \pi/2]$ and $h \in (0, \pi]$ the function Φ satisfies the condition

$$\frac{\pi}{\pi - 2}\int_0^1 \left(1 - \cos\frac{\pi h x}{2\tau\mu}\right)_* \left(1 + (\mu^2 - 1)\sin\frac{\pi x}{2}\right) dx \leq \frac{\Phi(h)}{\Phi(\tau)}, \tag{2}$$

where

$$(1 - \cos x)_* := \Big\{\ 1 - \cos x,\ \text{if}\ \ 0 < x \leq \pi;\ \ 2,\ \text{if}\ \ x \geq \pi\Big\},$$

then for any $n, r \in \mathbb{N},\ n > r \geq 1$ are true the equalities

$$b_n\left(W_q^{(r)}(\Phi; \mu); H_q\right) = d_n\left(W_q^{(r)}(\Phi; \mu); H_q\right) =$$

$$= E_n\left(W_q^{(r)}(\Phi; \mu); \mathscr{P}_{n-1}\right)_{H_q} = \frac{\pi}{2(\pi - 2)\alpha_{n,r}}\,\Phi\left(\frac{\pi}{2(n - r)\mu}\right), \tag{3}$$

where $\alpha_{n,r} = n(n-1)\cdots(n-r+1)$, $n \geq r$. Also in [15] proved that function $\varPhi(u) = u^{\alpha(\mu)}$, where

$$\alpha(\mu) = \frac{\pi^2}{2(\pi-2)\mu}\int_0^1 x\left(1+(\mu^2-1)\sin\frac{\pi x}{2}\right)\sin\frac{\pi x}{2\mu}dx,$$

satisfied the constraint (2), $\alpha(1) = 2/(\pi-2)$, $\lim\limits_{\mu\to\infty}\alpha(\mu) = 2$ and for all $\mu \in [1,\infty)$ satisfied the inequality $2/(\pi-2) \leq \alpha(\mu) \leq 2$. If the inequality (2) performs the change of variable $\tau = \pi/2(n-r)\mu$ $(n > r,\ 1/2 \leq \mu < \infty)$, then instead of (2) we obtain the equivalent condition

$$\frac{\pi}{\pi-2}\varPhi\left(\frac{\pi}{2(n-r)\mu}\right)\cdot\frac{1}{t}\int_0^t (1-\cos(n-r)x)_*\left(1+(\mu^2-1)\sin\frac{\pi x}{2t}\right)dx \leq \varPhi(t). \qquad (4)$$

The last inequality we will use in the proof of mentioned below theorem, in which the result (3) applies to the more general space $H_{q,\rho}$, $1 \leq q \leq \infty$, $0 < \rho \leq 1$.

Theorem 1. *Let $n, r \in \mathbb{N}$, $n > r$, $\mu \geq 1/2$ and majorant $\varPhi$ for any $h \in (0,\pi]$ satisfies the constraint (2). Then for all $1 \leq q \leq \infty$ and $0 < \rho \leq 1$ we have the equality*

$$b_n\left(W_q^{(r)}(\varPhi;\mu); H_{q,\rho}\right) = d_n\left(W_q^{(r)}(\varPhi;\mu); H_{q,\rho}\right) =$$

$$= E_n\left(W_q^{(r)}(\varPhi;\mu)\right)_{H_{q,\rho}} = \frac{\pi}{2(\pi-2)}\cdot\frac{\rho^n}{\alpha_{n,r}}\varPhi\left(\frac{\pi}{2(n-r)\mu}\right). \qquad (5)$$

Proof. In work [15, p. 93] proved that for any function $f \in H_q$, $1 \leq q \leq \infty$ and $u \in (0, \pi/(2n)]$, $n \in \mathbb{N}$

$$E_n(f) \leq \frac{\pi}{2u(\pi-2)}\int_0^u \omega_2(F;2x)_q\left\{1+\left[\left(\frac{\pi}{2un}\right)^2-1\right]\sin\frac{\pi x}{2u}\right\}dx \qquad (6)$$

and for functions of the form $f(z) = az^n$, $a \in \mathbb{C}$, $n \in \mathbb{N}$ the inequality (6) becomes equality. If in (6) we assume $\pi/(2un) = \mu$, where $un = \pi/(2\mu)$ $(\mu \geq 1/2)$, then the inequality (6) takes the form

$$E_n(f)_{H_q} \leq \frac{\mu n}{\pi-2}\int_0^{\pi/(2n\mu)} \omega_2(F;2x)_q\Big\{1+\left(\mu^2-1\right)\sin n\mu x\Big\}dx. \qquad (7)$$

According to the fact that for any functions $f \in H_q$, in which $f^{(r)} \in H_q$, holds the inequality [17, p. 287]

$$E_n(f)_{H_q} \leq \alpha_{n,r}^{-1}E_{n-r}(f^{(r)})_{H_q},\ n \geq r,\ n,r \in \mathbb{N},\ 1 \leq q \leq \infty, \qquad (8)$$

and takes into account the inequality (7), from (8) we obtain

$$E_n(f)_{H_q} \leq \frac{(n-r)\mu}{(\pi-2)\alpha_{n,r}}\int_0^{\pi/(2(n-r)\mu)} \omega_2(F^{(r)};2x)_q\Big\{1+\left(\mu^2-1\right)\sin(n-r)\mu x\Big\}dx.$$

Hence, for an arbitrary function $f \in W_q^{(r)}(\varPhi;\mu)$, according to the definition of the class, we have:

$$E_n(f)_{H_q} \leq \frac{\pi}{2(\pi-2)\alpha_{n,r}}\left(\frac{2(n-r)\mu}{\pi}\int_0^{\pi/(2(n-r)\mu)} \omega_2(F^{(r)};2x)_q\Big\{1+(\mu^2-1)\sin(n-r)\mu x\Big\}dx\right) \leq$$

$$\leq \frac{\pi}{2(\pi-2)\alpha_{n,r}}\, \Phi\left(\frac{\pi}{2(n-r)\mu}\right). \tag{9}$$

Since for an arbitrary function $f \in H_q$, $1 \leq q \leq \infty$ there holds the inequality [18, p. 49]

$$E_n(f)_{H_{q,\rho}} \leq \rho^n E_n(f)_q,\ 1 \leq q \leq \infty,\ 0 < \rho \leq 1,$$

then from (9) it is follows that for any $n > r$, $n, r \in \mathbb{N}$

$$E_n(f)_{H_{q,\rho}} \leq \frac{\pi\rho^n}{2(\pi-2)\alpha_{n,r}}\, \Phi\left(\frac{\pi}{2(n-r)\mu}\right). \tag{10}$$

From (10) and because of (1) we write

$$b_n\left(W_q^{(r)}(\Phi;\mu); H_{q,\rho}\right) \leq d_n\left(W_q^{(r)}(\Phi;\mu); H_{q,\rho}\right) \leq$$

$$\leq E_n\left(W_q^{(r)}(\Phi;\mu)\right)_{H_{q,\rho}} \leq \frac{\pi\rho^n}{2(\pi-2)\alpha_{n,r}}\, \Phi\left(\frac{\pi}{2(n-r)\mu}\right). \tag{11}$$

In order to obtain a lower bound specified in the n-widths of the set $\mathscr{P}_n \cap H_{q,\rho}$ we introduce the $(n+1)$-dimensional ball polynomials

$$\mathscr{B}_{n+1} := \left\{p_n \in \mathscr{P}_n :\ \|p_n\|_{H_{q,\rho}} \leq \frac{\pi\rho^n}{2(\pi-2)\alpha_{n,r}}\, \Phi\left(\frac{\pi}{2(n-r)\mu}\right)\right\}$$

and show that $\mathscr{B}_{n+1} \subset W_q^{(r)}(\Phi;\mu)$. Note that the inequality [7, p. 159]

$$\|p_n^{(r)}\|_{H_q} \leq \alpha_{n,r}\|p_n\|_q \quad (n > r,\ n, r \in \mathbb{N},\ 1 \leq q \leq \infty)$$

and the inequality proven by Pinkus [4, p. 255]

$$\|p_n\|_{H_q} \leq \rho^{-n}\|p_n\|_{q,\rho} \quad (1 \leq q \leq \infty,\ 0 < \rho \leq 1),$$

are true for an arbitrary polynomial $p_n \in \mathscr{P}_n$, we obtain

$$\|p_n^{(r)}\|_{H_q} \leq \alpha_{n,r}\rho^{-n}\|p_n\|_{q,\rho} \quad (n > r,\ 1 \leq q \leq \infty,\ 0 < \rho \leq 1). \tag{12}$$

Now, using the inequality [17, p. 291]

$$\omega_2(p_n; 2x)_q \leq 2(1-\cos nx)_*\|p_n\|_{H_q}, \tag{13}$$

replacing p_n with $p_n^{(r)}$ and then applying (12) for any polynomial $p_n \in \mathscr{B}_{n+1}$ we will have

$$\omega_2(p_n^{(r)}; 2x)_q \leq \frac{\pi}{(\pi-2)}\, \Phi\left(\frac{\pi}{2(n-r)\mu}\right)(1-\cos(n-r)x)_*. \tag{14}$$

In (14) for arbitrary $h \in (0, \pi]$, with a class definition $W_q^{(r)}(\Phi;\mu)$ and the constraint in (4), we have

$$\frac{1}{h}\int_0^h \omega_2(p_n^{(r)}; 2x)_q\left(1 + (\mu^2-1)\sin\frac{\pi x}{2h}\right)dx \leq$$

$$\leq \frac{\pi}{(\pi-2)}\, \Phi\left(\frac{\pi}{2(n-r)\mu}\right)\frac{1}{h}\int_0^h (1-\cos(n-r)x)_*\left(1 + (\mu^2-1)\sin\frac{\pi x}{2h}\right)dx \leq \Phi(h).$$

The last inequality means that the ball $\mathscr{B}_{n+1} \subset W_q^{(r)}(\Phi;\mu)$. Hence, as defined by Bernstein n-width, we obtain

$$b_n\left(W_q^{(r)}(\Phi;\mu); H_{q,\rho}\right) \geq b_n\left(\mathscr{B}_{n+1}; H_{q,\rho}\right) \geq \frac{\pi\rho^n}{2(\pi-2)\alpha_{n,r}}\, \Phi\left(\frac{\pi}{2(n-r)\mu}\right). \tag{15}$$

Comparing the upper bound (11) and the lower bound (15), we obtain the required equation (5), which completes the proof of Theorem 1. □

2 The Main Results

In order to find the exact values of the Gel'fand and linear n-widths it is important for us to construct the best linear method of approximation of functions of class $W_q^{(r)}(\varPhi;\mu)$ in the space $H_{q,\rho}$. With this purpose, for arbitrary analytic function in the unit disk $f(z)=\sum_{k=0}^{\infty}c_kz^k$ we write the following linear polynomial operator

$$\varLambda_{n-1,r,\rho}(f;z)\overset{def}{=}\sum_{k=0}^{r-1}c_k(f)z^k+$$

$$+\sum_{k=r}^{n-1}\left\{1+\frac{\alpha_{k,r}}{\alpha_{2n-k,r}}\rho^{2(n-k)}\left[\gamma_{k,r}\left(1-\left(\frac{k-r}{2n-k-r}\right)^2\right)-1\right]\right\}c_k(f)z^k \tag{16}$$

of degree $n-1$, where

$$\gamma_{k,r}\overset{def}{=}\frac{2\mu(n-r)}{\pi-2}\int\limits_0^{\pi/2\mu(n-r)}\cos(k-r)x\Big(1-\sin(n-r)\mu x\Big)dx,\ \ k\ge r\ge 1,\ k,r\in\mathbb{N}.$$

Theorem 2. *Let f be an arbitrary function in class $W_q^{(r)}(\varPhi;\mu)$, $r\in\mathbb{N}$, $1\le q\le\infty$, $\mu\ge 1/2$, $0<\rho\le 1$ and n is any positive integer greater than r. Then is holds the inequality*

$$\|f-\varLambda_{n-1,r,\rho}(f)\|_{H_{q,\rho}}\le\frac{\pi}{2(\pi-2)}\cdot\frac{\rho^n}{\alpha_{n,r}}\,\varPhi\left(\frac{\pi}{2(n-r)\mu}\right). \tag{17}$$

If majorized function $\varPhi$ for any $h\in(0,\pi]$ satisfies constraint (4), then inequality (17) cannot be improved in the sense that a function exists $f_0\in W_q^{(r)}(\varPhi;\mu)$, turning it into equality.

Proof. Let

$$\mathscr{L}_{n-1,r,\rho}(f;z)=\sum_{k=0}^{r-1}c_k(f)z^k++\sum_{k=r}^{n-1}\left(1-\frac{\alpha_{k,r}}{\alpha_{2n-k,r}}\rho^{2(n-k)}\right)c_k(f)z^k,$$

for any function $f\in W_q^{(r)}(\varPhi;\mu)$ we write the integral representation of the difference [7, p. 184-185], [13]

$$f(\rho z)-\mathscr{L}_{n-1,r,\rho}(f;\rho z)=\frac{\rho^nz^n}{2\pi}\int\limits_0^{2\pi}f^{(r)}(ze^{-it})\,\mathcal{K}_{n,r}(\rho,t)dt,\quad z\in U, \tag{18}$$

where

$$\mathcal{K}_{n,r}(\rho,t)\overset{def}{=}\frac{1}{\alpha_{n,r}}+2\sum_{j=1}^{\infty}\frac{\rho^j}{\alpha_{n+j,r}}\cos jt. \tag{19}$$

The representation (18) can be verified by direct calculation by expanding derivative $f^{(r)}$ in a Tailor series and integration by part of the resulting integrand.

Following the reasoning scheme of [7, p. 288] and [13, p. 325] we use the auxiliary function as an intermediate approximation of the function $f\in H_q$ with $f^{(r)}\in H_q$, which has the form

$$\mathscr{F}_t(f^{(r)};z)=\frac{\pi}{2t(\pi-2)}\int\limits_0^t\left\{f^{(r)}(ze^{ix})+f^{(r)}(ze^{-ix})\right\}\left(1-\sin\frac{\pi x}{2t}\right)dx. \tag{20}$$

If we put in (20) $t=t_*:=\pi/2(n-r)\mu$, $n>r$, $1/2\le\mu<\infty$ and expand derivatives $f^{(r)}$ in a Tailor series, we obtain

$$\mathscr{F}(F^{(r)};z):=\mathscr{F}_{t^*}(F^{(r)};z)=$$

$$= \frac{\mu(n-r)}{\pi-2} \int\limits_{0}^{\pi/2(n-r)\mu} \left\{F^{(r)}(ze^{ix}) + F^{(r)}(ze^{-ix})\right\}(1-\sin(n-r)\mu x)\,dx =$$

$$= \sum_{k=r}^{\infty} \gamma_{k,r}\alpha_{k,r}c_k(f)z^{k-r} := \sum_{k=0}^{\infty} \gamma_{k+r,r}\alpha_{k+r,r}c_{k+r}(f)z^k, \quad |z|<1, \tag{21}$$

which obviously is an element of H_q. For an arbitrary function $f \in H_q$, assuming

$$Q_{n-r-1,2}(f;z) \stackrel{def}{=} \sum_{k=0}^{n-r-1} \left(1 - \left(\frac{k}{2(n-r)-k}\right)^2\right) c_k(f)z^k$$

and, taking into account the form of the function (21), we write

$$Q_{n-r-1,2}(\mathscr{F}(F^{(r)});z) =$$

$$= \sum_{k=0}^{n-r-1} \gamma_{k+r,r}\,\alpha_{k+r,r}\,c_{k+r}(f)\left(1 - \left(\frac{k}{2(n-r)-k}\right)^2\right) z^k. \tag{22}$$

By symbol $\varphi_a^{(m)}$ we denote the derivative of mth ($m \in \mathbb{N}$) order of the function φ having the argument t of a complex number $z = \rho e^{it}$. Wherein

$$\varphi_a^{(1)}(z) = \frac{d\varphi(z)}{dz} \cdot \frac{\partial z}{\partial t} = \varphi'(z)zi, \quad \varphi_a^{(m)}(z) = \left\{\varphi_a^{(m-1)}(z)\right\}_a', \; m \geq 2,\, m \in \mathbb{N}.$$

It is known from [13, p. 325] that for any $z \in U$

$$\varphi(z) - Q_{n-r-1,2}(f;z) = -\frac{1}{2\pi}\int\limits_0^{2\pi} \varphi_a^{(2)}(ze^{-it})\,e^{i(n-r)t}\,G_{2,n-r}(t)dt, \tag{23}$$

where

$$G_{2,n-r}(t) \stackrel{def}{=} \frac{1}{(n-r)^2} + 2\sum_{k=j}^{\infty} \frac{\cos jt}{(n-r+j)^2}$$

is a non-negative integrable function [4, Lemma 2.2, p. 251]. From equation (23) using the generalized Minkowski inequality, we obtain

$$\left\|\varphi - Q_{n-r-1,2}(f)\right\|_{H_q} \leq \frac{1}{(n-r)^2}\,\|\varphi_a^{(2)}\|_{H_q}. \tag{24}$$

For an arbitrary function $f(z) \in W_q^{(r)}(\Phi;\mu)$ we construct a linear polynomial operator $(n-1)$th power of the following form

$$\Omega_{n-1,r,\rho}(f;\rho z) = \frac{\rho^n z^r}{2\pi}\int\limits_0^{2\pi} Q_{n-r-1,2}\Big(\mathscr{F}(F^{(r)});ze^{-it}\Big)e^{i(n-r)t}\,\mathcal{K}_{n,r}(\rho,t)dt =$$

$$= \sum_{k=r}^{n-1} \gamma_{k,r}\,\frac{\alpha_{k,r}}{\alpha_{2n-k,r}}\,\rho^{2(n-k)}\left(1 - \left(\frac{k-r}{2n-k-r}\right)^2\right) c_k(f)z^k, \tag{25}$$

the validity of which can be seen by considering the product of (19) and (22) and the subsequent term by term integration of the resulting integrand. Let

$$\varLambda_{n-1,r,\rho}(F;z) \stackrel{def}{=} \mathscr{L}_{n-1,r,\rho}(F;z) + \Omega_{n-1,r,\rho}(F;z)$$

and using the integral representation (18) and (25), for any $z \in U$ and $0 < \rho \le 1$ we write the equation

$$f(\rho z) - \Lambda_{n-1,r,\rho}(F;\rho z) =$$

$$= \frac{\rho^n z^r}{2\pi} \int_0^{2\pi} \Big\{ F^{(r)}(ze^{-it}) - Q_{n-r-1,2}\Big(\mathscr{F}(F^{(r)}); ze^{-it}\Big)\Big\} e^{i(n-r)t}\,\mathcal{K}_{n,r}(\rho,t)dt.$$

Hence, in view of the generalized Minkowski inequality we have

$$\Big\| f - \Lambda_{n-1,r,\rho}(f)\Big\|_{H_{q,\rho}} \le \frac{\rho^n}{\alpha_{n,r}} \Big\| F^{(r)} - Q_{n-r-1,2}\Big(\mathscr{F}(F^{(r)})\Big)\Big\|_{H_q} \le$$

$$\le \frac{\rho^n}{\alpha_{n,r}} \left\{ \Big\| F^{(r)} - \mathscr{F}(F^{(r)})\Big\|_q + \Big\| \mathscr{F}(F^{(r)}) - Q_{n-r-1,2}\Big(\mathscr{F}(F^{(r)})\Big)\Big\|_q \right\}. \tag{26}$$

We estimate the first term on the right hand side of (26), using again the above Minkowski inequality:

$$\Big\| F^{(r)} - \mathscr{F}(F^{(r)})\Big\|_q = \frac{(n-r)\mu}{\pi-2}\cdot$$

$$\cdot \left\| \int_0^{\pi/2(n-r)\mu} \Big\{ F^{(r)}(ze^{ix}) - 2F^{(r)}(z) + F^{(r)}(ze^{-ix})\Big\}\,(1-\sin(n-r)\mu x)\,dx \right\|_{H_q} \le$$

$$\le \frac{(n-r)\mu}{\pi-2} \int_0^{\pi/2(n-r)\mu} \omega_2(F^{(r)};2x)_q\,(1-\sin(n-r)\mu x)\,dx. \tag{27}$$

Putting $z = e^{it}$, we introduce the notation $f^{(r)}(ze^{\pm ix}) := G(t \pm x)$ and start to estimate the second term on the right hand side of (26), following the reasoning Taikov [17, p. 289], we assume that $f^{(r)}$ is an algebraic polynomial of degree p_m with $m \in \mathbb{N}$, since the set of all polynomials are dense in the space H_q. Obviously, with such an agreement the simultaneous approximation of functions and her derivative argument in H_q are valid, and therefore we can assume that for $m = 1, 2$, derivatives of $G^{(m)} \in H_q$. In this case, in view of (24) using (21) we obtain:

$$\Big\| \mathscr{F}(F^{(r)}) - Q_{n-r-1,2}\Big(\mathscr{F}(F^{(r)})\Big)\Big\|_{H_q} \le (n-r)^{-2} \Big\| \Big(\mathscr{F}(F^{(r)})\Big)_a^{(2)} \Big\|_{H_q} = \frac{\mu}{(n-r)(\pi-2)}\cdot$$

$$\cdot \left\| \int_0^{\pi/2(n-r)\mu} \Big\{ G^{(2)}(t+x) + G^{(2)}(t-x)\Big\}\,(1-\sin(n-r)\mu x)\,dx \right\|_{H_q}. \tag{28}$$

Performing double integration by parts on the right side obtained in (28), we write

$$\Big\| \mathscr{F}(F^{(r)}) - Q_{n-r-1,2}\Big(\mathscr{F}(F^{(r)})\Big)\Big\|_{H_q} \le$$

$$\le \frac{(n-r)\mu}{\pi-2} \left\| \int_0^{\pi/2(n-r)\mu} \Big\{ G(t+x) - 2G(t) + G(t-x)\Big\} \mu^2 \sin(n-r)\mu x dx \right\|_{H_q} \le$$

$$\le \frac{(n-r)\mu}{\pi-2} \int_0^{\pi/2(n-r)\mu} \omega_2(F^{(r)};2x)_q \mu^2 \sin(n-r)\mu x dx. \tag{29}$$

From (26) – (29) for an arbitrary function $f \in W_q^{(r)}(\Phi;\mu)$ taking into account the definition of the class, we have

$$\left\| f - \Lambda_{n-1,r,\rho}(f) \right\|_{H_{q,\rho}} \le \frac{\rho^n (n-r)\mu}{(\pi-2)\alpha_{n,r}} \cdot$$

$$\cdot \int_0^{\pi/2(n-r)\mu} \omega_2(F^{(r)};2x)_q \Big\{1 + (\mu^2-1)\sin(n-r)\mu x\Big\} dx = \frac{\pi\rho^n}{2(\pi-2)\alpha_{n,r}} \cdot$$

$$\cdot \left(\frac{2\mu(n-r)}{\pi} \int_0^{\pi/2(n-r)\mu} \omega_2(F^{(r)};2x)_q \Big\{1 + (\mu^2-1)\sin(n-r)\mu x\Big\} dx \right) \le$$

$$\le \frac{\pi}{2(\pi-2)} \cdot \frac{\rho^n}{\alpha_{n,r}} \Phi\left(\frac{\pi}{2(n-r)\mu}\right).$$

We show that the set of majorants satisfying the constraint (4) and belonging to the class $W_q^{(r)}(\Phi;\mu)$ is not empty. For this purpose, consider the following function

$$f_0(z) \overset{def}{=} \frac{\pi}{2(\pi-2)} \cdot \frac{1}{\alpha_{n,r}} \Phi\left(\frac{\pi}{2(n-r)\mu}\right) z^n, \quad n > r, \ \mu \ge 1/2$$

and show that f_0 belongs to the class $W_q^{(r)}(\Phi;\mu)$.

In the proof of Theorem 1, we have shown that $(n+1)$-dimensional sphere $\mathscr{B}_{n+1}$ polynomials $p_n \in \mathscr{P}_n$ with radius of not more than $\dfrac{\pi}{2(\pi-2)} \cdot \dfrac{\rho^n}{\alpha_{n,r}} \Phi\left(\dfrac{\pi}{2(n-r)\mu}\right)$ belongs to the class $W_q^{(r)}(\Phi;\mu)$, moreover majorant Φ satisfies the constraint (4). Since the norm of f_0 is equal to

$$\|f_0\|_{q,\rho} = \frac{\pi}{2(\pi-2)} \cdot \frac{\rho^n}{\alpha_{n,r}} \Phi\left(\frac{\pi}{2(n-r)\mu}\right),$$

the function f_0 belongs to $\mathscr{B}_{n+1}$ and therefore, $f_0 \in W_q^{(r)}(\Phi;\mu)$.

In according to the form of a linear operator (16), we have $\Lambda_{n-1,r,\rho}(f_0) \equiv 0$, and therefore

$$\left\| f_0 - \Lambda_{n-1,r,\rho}(f_0) \right\|_{H_{q,\rho}} = \left\| f_0 \right\|_{q,\rho} = \frac{\pi}{2(\pi-2)} \cdot \frac{\rho^n}{\alpha_{n,r}} \Phi\left(\frac{\pi}{2(n-r)\mu}\right), \tag{30}$$

Theorem 2 is proved. □

From the proved theorem 2 and equation (5) allows to formulate the following general statement.

Theorem 3. *If the majorant Φ satisfies the constraint (4), then for any $n, r \in \mathbb{N}$, $n > r$ and $0 < \rho \le 1$, $\mu \ge 1/2$ there hold the equalities*

$$\pi_n\left(W_q^{(r)}(\Phi;\mu); H_{q,\rho}\right) = E\left(W_q^{(r)}(\Phi;\mu), \mathscr{P}_{n-1}\right)_{H_{q,\rho}} =$$

$$= \mathscr{E}_n\left(W_q^{(r)}(\Phi;\mu); \Lambda_{n-1,r,\rho}\right)_{H_{q,\rho}} = \frac{\pi}{2(\pi-2)} \cdot \frac{\rho^n}{\alpha_{n,r}} \Phi\left(\frac{\pi}{2(n-r)\mu}\right), \tag{31}$$

where $\pi_n(\cdot)$ is any of the n-widths of $b_n(\cdot)$, $d_n(\cdot)$, $d^n(\cdot)$, $\delta_n(\cdot)$, and the best linear approximation method $\Lambda_{n-1,r,\rho}(\cdot)$ is defined by (16).

Proof. Using the definition of a linear n-width, from (17) we obtain the upper bound

$$\delta_n\left(W_q^{(r)}(\Phi;\mu); H_{q,\rho}\right) \le E\left(W_q^{(r)}(\Phi;\mu), \mathscr{P}_{n-1}\right)_{H_{q,\rho}} \le$$

$$\le \mathscr{E}_n\left(W_q^{(r)}(\Phi;\mu); \Lambda_{n-1,r,\rho}\right)_{H_{q,\rho}} = \frac{\pi}{2(\pi-2)} \cdot \frac{\rho^n}{\alpha_{n,r}} \Phi\left(\frac{\pi}{2(n-r)\mu}\right). \tag{32}$$

In view of the inequalities (1), between the above n-widths, comparing the upper bound (32), with (5) we obtain the required equality (31). It is also proved that the linear polynomial operator (15) is the best linear method of approximation of class $W_q^{(r)}(\Phi;\mu)$ $(r \in \mathbb{N},\ \mu \geq 1/2)$ in the space $H_{q,\rho}$ $(1 \leq q \leq \infty,\ 0 < \rho \leq 1)$. $\square$

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