Certain Analytic Functions with Missing Coefficients

Cai-Mei Yan[1*] and Jin-Lin Liu[2]

[1]Information Engineering College, Yangzhou University, Yangzhou, 225002, P.R. China
Email: cmyan@yzu.edu.cn
[2]Department of Mathematics, Yangzhou University, Yangzhou, 225002, P.R. China
Email: jlliu@yzu.edu.cn

Abstract Let $\mathcal{A}_n$ denote the class of functions of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$, which are analytic in the open unit disk $U = \{z : |z| < 1\}$. In this note we shall find $\max_{|z|=r<1} \mathrm{Re}\{f'(z) + \alpha z f''(z)\}$ under the condition $f'(z) \prec \frac{1+Az}{1+Bz}$ for $f \in \mathcal{A}_n$.

Keywords: Analytic function, subordination, missing coefficient.

1 Introduction

Throughout our present investigation, we assume that

$$n \in N, \ -1 \le B < 1, \ B < A, \ \alpha > 0 \ \text{ and } \ \beta < 1. \tag{1.1}$$

Let $\mathcal{A}_n$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \tag{1.2}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$.

For functions f and g analytic in U, we say that f is subordinate to g and write $f(z) \prec g(z) \quad (z \in U)$, if there exists an analytic function $w(z)$ in U such that

$$|w(z)| \le |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in U).$$

Furthermore, if the function g is univalent in U, then

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

In a recent paper [3], Gao and Zhou considered the following subclass of $\mathcal{A}_1$:

$$R(\beta, \alpha) = \{f \in \mathcal{A}_1 : \mathrm{Re}\{f'(z) + \alpha z f''(z)\} > \beta \quad (z \in U)\}.$$

Some interesting properties of the class $R(\beta, \alpha)$ have been given in [1]. For further information of the class $R(\beta, \alpha)$ one can see the related papers (see, e.g., [2,3,4,5,6,7,8,9]). Inspired by the above works, in this note we shall find

$$\max_{|z|=r<1} \mathrm{Re}\{f'(z) + \alpha z f''(z)\},$$

under the condition $f'(z) \prec \frac{1+Az}{1+Bz}$.

2 Main Results

Theorem 2.1. Let f belong to the class $\mathcal{A}_n$ and satisfy

$$f'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U). \tag{2.1}$$

Then

$$\operatorname{Re}\{f'(z)+\alpha z f''(z)\} \leq \frac{1+(A+B+n\alpha(A-B))r^n+ABr^{2n}}{(1+Br^n)^2} \quad \text{if } M_n(A,B,\alpha,r)\leq 0, \tag{2.2}$$

or

$$\operatorname{Re}\{f'(z)+\alpha z f''(z)\} \leq \frac{L_n^2-4\alpha^2K_AK_B}{4\alpha(A-B)r^{n-1}(1-r^2)K_B} \quad \text{if } M_n(A,B,\alpha,r)\geq 0, \tag{2.3}$$

where

$$\begin{cases} K_A = 1-A^2r^{2n}+nAr^{n-1}(1-r^2), \\ K_B = 1-B^2r^{2n}+nBr^{n-1}(1-r^2), \\ L_n = 2\alpha(1-ABr^{2n})+n\alpha(A+B)r^{n-1}(1-r^2)+(A-B)r^{n-1}(1-r^2), \\ M_n(A,B,\alpha,r) = 2\alpha K_B(1+Ar^n)-L_n(1+Br^n). \end{cases} \tag{2.4}$$

The result is sharp.

Proof. Equality in (2.2) occurs for $z=0$. Thus we assume that $0<|z|=r<1$. From (2.1) we can write

$$f'(z)=\frac{1+Az^n\varphi(z)}{1+Bz^n\varphi(z)} \quad (z\in U), \tag{2.5}$$

where $\varphi(z)$ is analytic and $|\varphi(z)|\leq 1$ in U. It follows from (2.5) that

$$\begin{aligned} f'(z)+\alpha z f''(z) &= f'(z)+\frac{\alpha(A-B)z^n(n\varphi(z)+z\varphi'(z))}{(1+Bz^n\varphi(z))^2} \\ &= f'(z)+\frac{n\alpha}{A-B}(A-Bf'(z))(f'(z)-1)+\frac{\alpha(A-B)z^{n+1}\varphi'(z)}{(1+Bz^n\varphi(z))^2}. \end{aligned} \tag{2.6}$$

With the help of the Carathéodory inequality:

$$|\varphi'(z)|\leq \frac{1-|\varphi(z)|^2}{1-r^2},$$

we obtain

$$\begin{aligned} \operatorname{Re}\left\{\frac{z^{n+1}\varphi'(z)}{(1+Bz^n\varphi(z))^2}\right\} &\leq \frac{r^{n+1}(1-|\varphi(z)|^2)}{(1-r^2)|1+Bz^n\varphi(z)|^2} \\ &= \frac{r^{2n}|A-Bf'(z)|^2-|f'(z)-1|^2}{(A-B)^2r^{n-1}(1-r^2)}. \end{aligned} \tag{2.7}$$

Put $f'(z)=u+iv \quad (u,v\in R)$. Then (2.6) and (2.7) provide

$$\begin{aligned} \operatorname{Re}\{f'(z)+\alpha z f''(z)\} &\leq \left(1+n\alpha\frac{A+B}{A-B}\right)u-\frac{n\alpha A}{A-B}-\frac{n\alpha B}{A-B}(u^2-v^2) \\ &\quad +\alpha\frac{r^{2n}((A-Bu)^2+(Bv)^2)-((u-1)^2+v^2)}{(A-B)r^{n-1}(1-r^2)} \\ &= \left(1+n\alpha\frac{A+B}{A-B}\right)u-\frac{n\alpha}{A-B}(A+Bu^2)+\alpha\frac{r^{2n}(A-Bu)^2-(u-1)^2}{(A-B)r^{n-1}(1-r^2)} \\ &\quad +\frac{\alpha}{A-B}\left(nB-\frac{1-B^2r^{2n}}{r^{n-1}(1-r^2)}\right)v^2. \end{aligned} \tag{2.8}$$

Note that

$$\begin{aligned} \frac{1-B^2r^{2n}}{r^{n-1}(1-r^2)} &\geq \frac{1-r^{2n}}{r^{n-1}(1-r^2)} = \frac{1}{r^{n-1}}(1+r^2+r^4+\cdots+r^{2(n-2)}+r^{2(n-1)}) \\ &= \frac{1}{2r^{n-1}}[(1+r^{2(n-1)})+(r^2+r^{2(n-2)})+\cdots+(r^{2(n-1)}+1)] \\ &\geq n\geq nB. \end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9) we get

$$\begin{aligned}\mathrm{Re}\{f'(z)+\alpha z f''(z)\} &\le \left(1+n\alpha\frac{A+B}{A-B}\right)u-\frac{n\alpha}{A-B}(A+Bu^2)+\alpha\frac{r^{2n}(A-Bu)^2-(u-1)^2}{(A-B)r^{n-1}(1-r^2)}\\ &=\psi_n(u) \quad \text{(say)}.\end{aligned} \tag{2.10}$$

It is well known that for $|\xi|\le\sigma \quad (\sigma<1)$,

$$\left|\frac{1+A\xi}{1+B\xi}-\frac{1-AB\sigma^2}{1-B^2\sigma^2}\right|\le\frac{(A-B)\sigma}{1-B^2\sigma^2} \tag{2.11}$$

and

$$\frac{1-A\sigma}{1-B\sigma}\le\mathrm{Re}\left\{\frac{1+A\xi}{1+B\xi}\right\}\le\frac{1+A\sigma}{1+B\sigma}. \tag{2.12}$$

Also (2.5) and (2.12) imply that

$$\frac{1-Ar^n}{1-Br^n}\le\mathrm{Re}\{f'(z)\}\le\frac{1+Ar^n}{1+Br^n}.$$

Let us now calculate the maximum value of $\psi_n(u)$ on the segment $\left[\frac{1-Ar^n}{1-Br^n},\frac{1+Ar^n}{1+Br^n}\right]$. Obviously,

$$\psi_n'(u)=1+n\alpha\frac{A+B}{A-B}-\frac{2n\alpha B}{A-B}u+2\alpha\frac{(1-ABr^{2n})-(1-B^2r^{2n})u}{(A-B)r^{n-1}(1-r^2)},$$

$$\psi_n''(u)=-\frac{2\alpha}{A-B}\left(nB+\frac{1-B^2r^{2n}}{r^{n-1}(1-r^2)}\right)<0 \quad \text{(see (2.9))} \tag{2.13}$$

and $\psi_n'(u)=0$ if and only if

$$\begin{aligned}u=u_n&=\frac{2\alpha(1-ABr^{2n})+n\alpha(A+B)r^{n-1}(1-r^2)+(A-B)r^{n-1}(1-r^2)}{2\alpha[1-B^2r^{2n}+nBr^{n-1}(1-r^2)]}\\ &=\frac{L_n}{2\alpha K_B} \quad \text{(see (2.4))}.\end{aligned} \tag{2.14}$$

Since

$$\begin{aligned}&2\alpha K_B(1-Ar^n)-L_n(1-Br^n)\\ &=2\alpha[(1-Ar^n)(1-B^2r^{2n})-(1-Br^n)(1-ABr^{2n})]\\ &\quad -n\alpha r^{n-1}(1-r^2)[(A+B)(1-Br^n)-2B(1-Ar^n)]-(A-B)r^{n-1}(1-r^2)(1-Br^n)\\ &=-2\alpha(A-B)r^n(1-Br^n)-n\alpha(A-B)r^{n-1}(1-r^2)(1+Br^n)-(A-B)r^{n-1}(1-r^2)(1-Br^n)\\ &<0,\end{aligned}$$

we see that

$$u_n>\frac{1-Ar^n}{1-Br^n}\ . \tag{2.15}$$

But u_n is not always less than $\frac{1+Ar^n}{1+Br^n}$. The following two cases arise.

Case (i). $u_n\ge\frac{1+Ar^n}{1+Br^n}$, that is, $M_n(A,B,\alpha,r)$ (given by (2.4))≤ 0. In view of $\psi_n'(u_n)=0$ and (2.13), the function $\psi_n(u)$ is increasing on the segment $\left[\frac{1-Ar^n}{1-Br^n},\frac{1+Ar^n}{1+Br^n}\right]$. Therefore we deduce from (2.10) that, if $M_n(A,B,\alpha,r) \le 0$, then

$$\begin{aligned}\mathrm{Re}\{f'(z)+\alpha z f''(z)\} &\le \psi_n\left(\frac{1+Ar^n}{1+Br^n}\right)\\ &= \left(1+n\alpha\frac{A+B}{A-B}\right)\left(\frac{1+Ar^n}{1+Br^n}\right)-\frac{n\alpha}{A-B}\left(A+B\left(\frac{1+Ar^n}{1+Br^n}\right)^2\right)\\ &= \frac{1+Ar^n}{1+Br^n}-\frac{n\alpha}{A-B}\left(1-\frac{1+Ar^n}{1+Br^n}\right)\left(A-B\frac{1+Ar^n}{1+Br^n}\right)\\ &= \frac{1+(A+B+n\alpha(A-B))r^n+ABr^{2n}}{(1+Br^n)^2}\ .\end{aligned}$$

This proves (2.2).

Next we consider the function f defined by

$$f(z)=\int_0^z \frac{1+At^n}{1+Bt^n}\mathrm{d}t$$

which satisfies the condition (2.1). It is easy to check that

$$f'(r)+\alpha r f''(r)=\frac{1+(A+B+n\alpha(A-B))r^n+ABr^{2n}}{(1+Br^n)^2},$$

which shows that the inequality (2.2) is sharp.

Case (ii). $u_n \le \frac{1+Ar^n}{1+Br^n}$, that is, $M_n(A,B,\alpha,r)\ge 0$. In this case we easily have

$$\mathrm{Re}\{f'(z)+\alpha z f''(z)\} \le \psi_n(u_n). \tag{2.16}$$

In view of (2.4), $\psi_n(u)$ in (2.10) can be written as

$$\psi_n(u)=\frac{-\alpha K_B u^2+L_n u-\alpha K_A}{(A-B)r^{n-1}(1-r^2)}. \tag{2.17}$$

Therefore, if $M_n(A,B,\alpha,r)\ge 0$, then it follows from (2.14), (2.16) and (2.17) that

$$\begin{aligned}\mathrm{Re}\{f'(z)+\alpha z f''(z)\} &\le \frac{-\alpha K_B u_n^2+L_n u_n-\alpha K_A}{(A-B)r^{n-1}(1-r^2)}\\ &= \frac{L_n^2-4\alpha^2 K_A K_B}{4\alpha(A-B)r^{n-1}(1-r^2)K_B}.\end{aligned}$$

To show that the inequality (2.3) is sharp, we take

$$f(z)=\int_0^z \frac{1+At^n\varphi(t)}{1+Bt^n\varphi(t)}\mathrm{d}t \quad \text{and} \quad \varphi(z)=\frac{z-c_n}{1-c_n z}$$

where $c_n \in R$ is determined by

$$f'(r)=\frac{1+Ar^n\varphi(r)}{1+Br^n\varphi(r)}=u_n\in\left(\frac{1-Ar^n}{1-Br^n},\frac{1+Ar^n}{1+Br^n}\right].$$

Clearly, $-1<\varphi(r)\le 1, -1\le c_n<1, |\varphi(z)|\le 1$ $(z\in U)$, and so f satisfies the condition (2.1). Since

$$\varphi'(r)=\frac{1-c_n^2}{(1-c_n r)^2}=\frac{1-|\varphi(r)|^2}{1-r^2},$$

from the above argument we find that

$$f'(r)+\alpha r f''(r)=\psi_n(u_n).$$

Now the proof of the theorem is completed.

Corollary 2.2. Let f belong to the class $\mathcal{A}_1$ and satisfy $\mathrm{Re}\{f'(z)\} > \beta$ $(\beta < 1; z \in U)$. Then for $|z| = r < 1$,

$$\mathrm{Re}\left\{f'(z) + \alpha z f''(z)\right\} \leq \beta + (1-\beta)\frac{1+2\alpha r - r^2}{(1-r)^2}.$$

The result is sharp.

Proof. By considering $\frac{f'(z)-\beta}{1-\beta}$ instead of $f'(z)$, we only need to prove the corollary for $\beta = 0$. Setting $n = A = 1$ and $B = -1$ in (2.4), we get

$$K_1 = 2(1-r^2), \quad K_{-1} = 0, \quad L_1 = 2\alpha(1+r^2) + 2(1-r^2)$$

and

$$M_1(1,-1,\alpha,r) = -2(1-r)[1+\alpha-(1-\alpha)r^2] \leq 0.$$

Consequently, an application of (2.2) in Theorem 2.1 yields

$$\mathrm{Re}\left\{f'(z) + \alpha z f''(z)\right\} \leq \frac{1+2\alpha r - r^2}{(1-r)^2}.$$

Furthermore the sharpness follows immediately from that of Theorem 2.1.

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