

Fixed Point Approximations with Finite Relatively Nonexpansive Maps in Some Real Banach Spaces

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Abstract Let E be a uniformly smooth and uniformly convex real Banach space and C be a nonempty, closed and convex subset of E . In this paper, it is shown that $\{x_n\}$ obtained from Batsari's[21] CQ algorithm with relatively nonexpansive maps converges strongly to a point \hat{x} which is also a common fixed point of some finite relatively nonexpansive mappings and solves a system of equilibrium problems in E . The result obtained improves some existing results in the literature.

Keywords: relatively nonexpansive mappings; equilibrium problems; uniformly smooth space; uniformly convex space; bounded sequence.

1 Introduction

Let E be a real Banach space and E^* be its dual Space. Let C be a nonempty, closed and convex subset of E . Let $\{f_k\}$ for $k \in \Gamma$ be a family of bifunctions from $C \times C$ to \mathbb{R} , where \mathbb{R} is the set of real numbers and Γ is an arbitrary index set. The equilibrium problems are to find $\hat{x} \in C$ such that,

$$f_k(\hat{x}, y) \geq 0, \quad \forall y \in C \text{ and } k \in \Gamma. \quad (1)$$

If Γ is a singletone then, problem (1) becomes the following equilibrium problem of finding $\hat{x} \in C$ such that,

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \quad (2)$$

The solution set of (1) and (2) are denoted by $EP(f_k)$ and $EP(f)$ respectively.

Note: For solving the equilibrium problems (1) we assume that, $\forall k \in \Gamma$ the bi-function f_k satisfies the following conditions:

(A1) $f_k(x, x) = 0, \quad \forall x \in C$.

(A2) f_k is monotone, i.e $f_k(x, y) + f_k(y, x) \leq 0$, for any $x, y \in C$.

(A3) For each $x, y, z \in C$,

$$\limsup_{t \rightarrow 0} f_k(tz + (1-t)x, y) \leq f_k(x, y).$$

(A4) $f_k(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

Let X, Y be sets such that $X \subset Y$, consider a map $S : X \rightarrow Y$, a set $F(S) = \{z \in X : Sz = z\}$ is called the fixed point set of S .

A mapping $S : C \rightarrow E$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Let E be a real Banach space and E^* be its dual, for all $x \in E$ and $x^* \in E^*$ we denote the value of x^* at x by $\langle x, x^* \rangle$. So, the duality mapping J on E is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1 \quad \forall x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$.

A Banach space E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$ there exists $\delta > 0$ such that

$$\frac{\|x+y\|}{2} \leq 1 - \delta, \forall x, y \in E \text{ with } \|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon.$$

A Banach space E is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists $\forall x, y \in S(E)$ where $S(E) = \{z \in E : \|z\| = 1\}$. Also, the space E is said to be uniformly smooth if the above limit exists uniformly for $x, y \in S(E)$.

The following definition can be found in [11]. Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty, closed and convex subset of E , define a function $\phi : E \times E \rightarrow \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

It is obvious from the definition of $\phi(x, y)$ that,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in C. \quad (3)$$

Let E be a smooth, strictly convex and reflexive Banach space, C be a nonempty, closed and convex subset of E , $S : C \rightarrow C$ be a mapping and $F(S)$ be the set of fixed points of S . A point $p \in C$ is said to be an asymptotic fixed point of S [15] if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightarrow p$ and $\|x_n - Sx_n\| \rightarrow 0$. The set of all asymptotic fixed points of S is denoted by $\tilde{F}(S)$.

A mapping $S : C \rightarrow C$ is said to be relatively nonexpansive[15] if

1. $F(S) \neq \emptyset$.
2. $F(S) = \tilde{F}(S)$.
3. $\phi(p, Sy) \leq \phi(p, y), \quad \forall y \in C, p \in F(S)$.

From Alber[17], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg \min_{y \in C} \phi(y, x), \quad \forall x \in E.$$

A Banach space E is said to have a Kadec-Klee property[15] if for every sequence $\{x_n\}$ in E , $x_n \rightarrow x$ and $\|x_n\| \rightarrow \|x\| \implies \|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

Remark : The following basic property is true: We can find in Cioranescu [7] that, if E is uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E .

The problem (1) is very general in the sense that, it includes as special cases: optimization problems, variational inequalities, mini-max problems, Nash equilibrium problem in noncooperative games and others; see for example [4,11]. Various methods have been proposed to solve problem (1).

In 1953 Mann[18] introduced a well known classical iterative process to approximate a fixed point of a nonexpansive mapping. The iterative process is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(x_n) \quad n \geq 0, \quad (4)$$

where initial element x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfies

1. $\lim_{n \rightarrow \infty} \alpha_n = 0$.
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$.

But, Mann's iterative process has only weak convergence, even in a Hilbert space. Therefore, many authors try to modify Mann's iterative process in order to have strong convergence, the so called hybrid projection iterative method is one of such modifications.

The hybrid projection iterative algorithm(HPIA) was introduced initially by Haugazeau[19] in 1968. For over 40 years, HPIA has received rapid developments.

In 2006, Nakajo et al.[19] proposed the following modification of Mann's iterative scheme for a nonexpansive mapping T in a real Hilbert space H :

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \quad (5)$$

where C is a closed, convex subset of H , $P_{C_n \cap Q_n}$ denotes the metric projection from H onto a nonempty, closed and convex subset $C_n \cap Q_n$ of H . They proved that, if the sequence $\{\alpha_n\}$ is bounded above by one then, the sequence $\{x_n\}$ generated by (5) converges strongly to $P_{F(T)}(x_0)$, where, $F(T)$ is the fixed point set of T .

In 2009, Takahashi and Zembayashi[15] proposed the following modification of Mann’s iterative scheme for relatively nonexpansive mapping S , in a real uniformly convex and uniformly smooth Banach space E :

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x, \end{cases} \tag{6}$$

where J is the duality mapping on E and f is a bifunction from $C \times C$ to \mathbb{R} satisfying conditions (A1)-(A4). They proved that, if $\liminf_{n \rightarrow \infty} (1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$ then, the sequence $\{x_n\}$ generated by (6) converges strongly to $\Pi_{F(S) \cap EP(f)} x$, where $\Pi_{F(S) \cap EP(f)}$ is the generalized projection of E onto $F(S) \cap EP(f)$.

Duan and Zhao[8] studied a new hybrid methods for equilibrium problems and strictly pseudocontractions.

In 2010 P. Duan[9] proposed the following iterative scheme for some finite family of strictly pseudocontraction mappings and systems of equilibrium problems in a real Hilbert space H :

$$\begin{cases} u_n = T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \dots T_{r_{2,n}}^{f_2} T_{r_{1,n}}^{f_1} x_n, \\ A_n^{\lambda_n} = \lambda_n I + (1 - \lambda_n)A_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)A_n^{\lambda_n} u_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \end{cases} \tag{7}$$

where $\{\alpha_n\} \subset [0, a]$, $\{\lambda_n\} \subset [l, b]$ for some $a, l \in [0, 1)$, $b \in [l, 1)$ and $\{r_{k,n}\} \subset (0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_{k,n} > 0$ for all $k \in \{1, 2, \dots, M\}$. Then, she proved that, the sequence $\{x_n\}$ generated by (7) converges strongly to $P_{\Omega} x_1$, where $\Omega = \bigcap_{i=1}^N F(S_i) \cap (\bigcap_{K=1}^M EP(f_k))$. She also proved a strong convergence using cyclic algorithm in the same journal article.

Recently in 2014, U.Y. Batsari[21] proposed the following iterative scheme for some finite family of ϕ -nonexpansive mappings and systems of equilibrium problems in a real Banach space E :

$$\begin{cases} y_n = j^{-1}(\alpha_n jx_n + (1 - \alpha_n)jA_N^{\lambda_n} x_n), \\ u_n = T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \dots T_{r_{2,n}}^{f_2} T_{r_{1,n}}^{f_1} y_n, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, jx_1 - jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_1, \end{cases} \tag{8}$$

for every $n \in \mathbb{N}$, where $j \in J$ and J is the duality mapping on E , $\{\alpha_n\} \subset [0, a]$ such that $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ for some $a \in [0, 1)$, $\{\lambda_n\} \subset [0, 1)$, and $\{r_{k,n}\} \subset (0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_{k,n} > 0$ for all $k \in \{1, 2, 3, \dots, M\}$ then, $\{x_n\}$ generated by (8) converges strongly to $\Pi_{\Omega} x_1$.

In this paper, motivated by [15] and [21], a strong convergence theorem for approximating common fixed points for finite family of relatively nonexpansive maps and common solution of finite family of equilibrium problems in a uniformly convex and uniformly smooth real Banach space E is studied; which is an improvement to [15] that consider only one relatively nonexpansive map and one equilibrium problem. Also, the importance of relatively nonexpansive maps especially the resolvent of maximal monotone operators inspires the researcher to substitute ϕ -nonexpansive maps with relatively nonexpansive maps as used in [21].

2 Preliminaries

We shall make use of the following results.

Lemma 2.1 [17]. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E then,*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C \text{ and } y \in E.$$

Lemma 2.2 [11]. *Let E be a smooth and uniformly convex Banach space, let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded, if $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$ then, $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Lemma 2.3 [6]. *Let E be a uniformly convex Banach space and $r > 0$ then, there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that, $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|),$$

for all $x, y \in B_r$ and $t \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.4 [15]. *Let C be a nonempty, closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach space E . Let f be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4). For $r > 0$ define a function $T_r : E \rightarrow C$ by*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\},$$

for all $x \in E$ then, the following holds

1. T_r is single valued.
2. T_r is firmly nonexpansive type, i.e for all $x, y \in E$

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle.$$

3. $F(T_r) = EP(f)$.
4. $EP(f)$ is closed and convex.
5. T_r is relatively nonexpansive.

Now, in view of the above lemma define the maps $T_{r_k, n}^{f_k} : E \rightarrow C$ for $k \in \{1, 2, \dots, M\}$ by $T_{r_k, n}^{f_k}(x) = \{z \in C : f_k(z, y) + \frac{1}{r_k, n}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\} \quad \forall x \in E, \quad n \in \mathbb{N}$ and $\{r_{k, n}\} \subseteq (0, \infty)$. Let $\Theta_n^k = T_{r_k, n}^{f_k} T_{r_{k-1, n}}^{f_{k-1}} \dots T_{r_{2, n}}^{f_2} T_{r_{1, n}}^{f_1}$ then,

1. If $h \in \bigcap_{k=1}^M EP(f_k)$ we have $\Theta_n^k h = h, \quad \forall k \in \{1, 2, \dots, M\}$.
2. Θ_n^M is relatively nonexpansive hence, $\phi(h, \Theta_n^M x) \leq \phi(h, x), \quad \forall h \in F(\Theta_n^M)$ and $x \in C$.

Lemma 2.5 [15]. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E , for $x \in E$ and $q \in F(T_r)$ we have*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma 2.6 [17]. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E , let $x \in E$ and $z \in C$ then,*

$$z = \Pi_C x \iff \langle y - z, Jx - Jz \rangle \leq 0, \quad \forall y \in C.$$

Also, for $x, y \in E$,

$$\phi(x, y) = 0 \iff x = y$$

Lemma 2.7. *Let E be a real Banach Space. Given a Natural number N , assume for each $1 \leq i \leq N$, S_i is a well defined mapping from a subset of E to E . Let $A_n = \sum_{i=1}^N \alpha_i^{(n)} S_i$ and $A_n^{\lambda_n} = \lambda_n I + (1 - \lambda_n) A_n$, where $\{\alpha_i^{(n)}\}_{i=1}^N$ is a sequence of positive real numbers such that $\sum_{i=1}^N \alpha_i^{(n)} = 1 \quad \forall n \in \mathbb{N}$, suppose $\{S_i\}_{i=1}^N$ has a common fixed point $\forall N \in \mathbb{N}$ then, $\bigcap_{i=1}^N F(S_i) = F(A_n) = F(A_n^{\lambda_n}), \quad \forall \lambda_n \in (0, 1)$.*

Proof. To prove the lemma, it is enough to show that the result holds for $N = 2$. Now, let $N = 2$,

First Claim: $\bigcap_{i=1}^2 F(S_i) \subseteq F(A_n)$.

Let $x \in \bigcap_{i=1}^2 F(S_i)$ then $x = S_i x$ for $i = 1, 2$. So,

$$\begin{aligned} A_n x &= (\alpha_1^{(n)} S_1 + \alpha_2^{(n)} S_2)x \\ &= \alpha_1^{(n)} S_1 x + \alpha_2^{(n)} S_2 x \\ &= \alpha_1^{(n)} x + \alpha_2^{(n)} x \\ &= (\alpha_1^{(n)} + \alpha_2^{(n)})x \\ &= x. \end{aligned}$$

Thus, $\bigcap_{i=1}^2 F(S_i) \subseteq F(A_n)$.

Conversely, let $x \in F(A_n)$ then,

$$\begin{aligned} 0 &= \|x - A_n x\| \\ &= \|x - (\alpha_1^{(n)} S_1 x + \alpha_2^{(n)} S_2 x)\| \\ &= \|(\alpha_1^{(n)} + \alpha_2^{(n)})x - (\alpha_1^{(n)} S_1 x + \alpha_2^{(n)} S_2 x)\| \\ &= \|(\alpha_1^{(n)} x - \alpha_1^{(n)} S_1 x) - (\alpha_2^{(n)} S_2 x - \alpha_2^{(n)} x)\| \\ &\geq \left| \|\alpha_1^{(n)} x - \alpha_1^{(n)} S_1 x\| - \|\alpha_2^{(n)} S_2 x - \alpha_2^{(n)} x\| \right|, \end{aligned}$$

this implies that,

$$0 = \left| \alpha_1^{(n)} \|x - S_1 x\| - \alpha_2^{(n)} \|S_2 x - x\| \right|,$$

which means that,

$$0 = \alpha_1^{(n)} \|x - S_1 x\| - \alpha_2^{(n)} \|S_2 x - x\|, \quad \forall n \in \mathbb{N},$$

from which we have

$$\alpha_1^{(n)} \|x - S_1 x\| = \alpha_2^{(n)} \|x - S_2 x\|, \quad \forall n \in \mathbb{N}.$$

That is $x = S_1 x$ and $x = S_2 x$. So, $x \in \bigcap_{i=1}^2 F(S_i)$ and thus, $F(A_n) \subseteq \bigcap_{i=1}^2 F(S_i)$. Therefore, $\bigcap_{i=1}^2 F(S_i) = F(A_n)$.

Second Claim: $F(A_n) = F(A_n^{\lambda_n})$.

Let $x \in F(A_n)$ then,

$$\begin{aligned} A_n^{\lambda_n} x &= \lambda_n x + (1 - \lambda_n) A_n x \\ &= (\lambda_n + (1 - \lambda_n))x \\ &= x, \end{aligned}$$

which means, $F(A_n) \subseteq F(A_n^{\lambda_n})$.

Conversely, let $x \in F(A_n^{\lambda_n})$,

$$\begin{aligned} 0 &= \|x - A_n^{\lambda_n} x\| \\ &= \|x - (\lambda_n x + (1 - \lambda_n) A_n x)\| \\ &= \|(1 - \lambda_n)x - (1 - \lambda_n) A_n x\| \\ &= (1 - \lambda_n) \|x - A_n x\|. \end{aligned}$$

Therefore, $A_n x = x$ and so, $F(A_n^{\lambda_n}) \subseteq F(A_n)$. Hence, $F(A_n) = F(A_n^{\lambda_n})$. □

3 Main Result

Let E be a uniformly smooth and uniformly convex real Banach space. Let C be a nonempty, closed and convex subset of E . Let f_k for $k \in \{1, 2, 3, \dots, M\}$, be a bifunction from $C \times C$ to \mathbb{R} which satisfies conditions (A1)-(A4) and $f_k(x, y) \leq f_{k+1}(x, y) \quad \forall x, y \in C$. Let $V = \{\alpha_i S_i : C \rightarrow C \mid i = 1, 2, \dots, N \text{ and } \alpha_i \in (0, 1]\}$ be a convex set of relatively nonexpansive maps and contain identity map I . Assume that, $\Omega = \bigcap_{i=1}^N F(S_i) \cap (\bigcap_{k=1}^M EP(f_k)) \neq \emptyset$. Assume also that $\{\eta_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all $n \in \mathbb{N}$ and $\inf_{n \geq 1} \eta_i^{(n)} > 0$ for all $i, 1 \leq i \leq N$. Let the mapping $A_N^{\lambda_n}$ be defined by

$$A_N^{\lambda_n} = \lambda_n I + (1 - \lambda_n) \sum_{i=1}^N \eta_i^{(n)} S_i.$$

Given $x_1 \in C$, let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences which are generated by the following algorithm:

$$\begin{cases} y_n = j^{-1}(\alpha_n j x_n + (1 - \alpha_n) j A_N^{\lambda_n} x_n), \\ u_n = T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \dots T_{r_{2,n}}^{f_2} T_{r_{1,n}}^{f_1} y_n, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, j x_1 - j x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_1, \end{cases}$$

for every $n \in \mathbb{N}$, where $j \in J$ and J is the duality mapping on E , $\{\alpha_n\} \subset [0, a]$ such that $\liminf_{n \rightarrow \infty} \alpha_n (1 - \alpha_n) > 0$ for some $a \in [0, 1)$, $\{\lambda_n\} \subset [0, 1)$, and $\{r_k, n\} \subset (0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_{k,n} > 0$ for all $k \in \{1, 2, 3, \dots, M\}$ then, $\{x_n\}$ converges strongly to $\Pi_{\Omega} x_1$.

Proof. Let us denote $\Theta_n^k = T_{r_{k,n}}^{f_k} T_{r_{k-1,n}}^{f_{k-1}} \dots T_{r_{2,n}}^{f_2} T_{r_{1,n}}^{f_1}$ for all $n \in \mathbb{N}$ and $k \in \{1, 2, 3, \dots, M\}$, then $u_n = \Theta_n^M y_n$.

We start by showing that $\{x_n\}$ is well defined.

To prove this, it is enough to show $C_n \cap Q_n$ is nonempty, closed and convex subset of $E \quad \forall n \in \mathbb{N}$. Clearly C_n, Q_n are closed and Q_n is convex. Also,

$$\phi(z, u_n) \leq \phi(z, x_n) \iff \|u_n\|^2 - \|x_n\|^2 - 2\langle z, j u_n - j x_n \rangle \leq 0.$$

This implies that, C_n is convex. So, $C_n \cap Q_n$ is closed and convex.

Next, we show that $C_n \cap Q_n \neq \emptyset$.

We are to show $\Omega \subset C_n \cap Q_n$. As $\Omega = \bigcap_{i=1}^N F(S_i) \cap (\bigcap_{k=1}^M EP(f_k)) \neq \emptyset$. Let $p \in \Omega$ then,

$$\begin{aligned} \phi(p, u_n) &= \phi(p, \Theta_n^M y_n) \\ &\leq \phi(p, y_n) \\ &= \phi(p, j^{-1}(\alpha_n j x_n + (1 - \alpha_n) j A_N^{\lambda_n} x_n)) \\ &= \|p\|^2 - 2\langle p, \alpha_n j x_n + (1 - \alpha_n) j A_N^{\lambda_n} x_n \rangle + \|\alpha_n j x_n + (1 - \alpha_n) j A_N^{\lambda_n} x_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n \langle p, j x_n \rangle - 2(1 - \alpha_n) \langle p, j A_N^{\lambda_n} x_n \rangle + \alpha_n \|x_n\|^2 + (1 - \alpha_n) \|A_N^{\lambda_n} x_n\|^2 \\ &= \alpha_n \phi(p, x_n) + (1 - \alpha_n) \phi(p, A_N^{\lambda_n} x_n) \\ &\leq \phi(p, x_n). \end{aligned}$$

Therefore, $p \in C_n$ and this implies that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. Next, we show by induction that, $\Omega \subset C_n \cap Q_n$.

From $Q_1 = C$ we have $\Omega \subset C_1 \cap Q_1$. Suppose that, $\Omega \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$ then, there exists $x_{k+1} \in C_k \cap Q_k$ such that,

$$x_{k+1} = \Pi_{Q_k \cap C_k} x_1.$$

But, from the definition of x_{k+1} we have

$$\langle x_{k+1} - z, j x_1 - j x_{k+1} \rangle \geq 0, \quad \forall z \in Q_k \cap C_k.$$

Since, $\Omega \subset Q_k \cap C_k$ then, $\forall z \in \Omega$ we have

$$\langle x_{k+1} - z, jx_1 - jx_{k+1} \rangle \geq 0,$$

which implies that, $z \in Q_{k+1}$ from the definition of Q_n in the algorithm given above. So, we have

$$\Omega \subset Q_{k+1},$$

for $\Omega \subset C_n \forall n \in \mathbb{N}$ we write

$$\Omega \subset Q_{k+1} \cap C_{k+1},$$

which means that, $\Omega \subset Q_n \cap C_n$ for all $n \in \mathbb{N}$. Hence, $\{x_n\}$ is well defined.

From the definition of Q_n , we have $x_n = \Pi_{Q_n} x_1$. Using lemma 2.1 we have

$$\begin{aligned} \phi(x_n, x_1) &= \phi(\Pi_{Q_n} x_1, x_1) \\ &\leq \phi(p, x_1) - \phi(p, \Pi_{Q_n} x_1) \\ &\leq \phi(p, x_1), \quad \forall p \in \Omega \subset Q_n. \end{aligned}$$

Thus, $\{\phi(x_n, x_1)\}$ is bounded. Therefore, $\{x_n\}$ and $\{S_i x_n\}$ are bounded for all $1 \leq i \leq N$. Since $x_{n+1} = \Pi_{Q_n \cap C_n} x_1 \in Q_n \cap C_n$ and $x_n = \Pi_{Q_n} x_1$ from the definition of Π_{Q_n} we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \in \mathbb{N}.$$

Thus, $\{\phi(x_n, x_1)\}$ is nondecreasing. Therefore, its limit exists. From $x_n = \Pi_{Q_n} x_1$ and lemma 2.1 we also have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{Q_n} x_1) \\ &\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{Q_n} x_1, x_1) \\ &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1), \quad \forall n \in \mathbb{N}. \end{aligned}$$

This means that, $\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{Q_n \cap C_n} x_1 \in C_n$ we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, u_n) = 0.$$

Since $\lim_{n \rightarrow \infty} \phi(x_{n+1}, u_n) = \lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0$, with E uniformly convex and smooth, from lemma 2.2 we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0.$$

So, it follows that

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0.$$

Since J is norm-to-norm uniformly continuous on bounded sets and $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$, we have

$$\lim_{n \rightarrow \infty} \|jx_n - ju_n\| = 0.$$

Let $r = \sup_{n \in \mathbb{N}} \{\|x_n\|, \|S_i x_n\|\}$ for $1 \leq i \leq N$. Since E is a uniformly smooth Banach space, we know that, E^* is a uniformly convex Banach space. Therefore, from lemma 2.3, there exists a continuous, strictly increasing and convex function g , with $g(0) = 0$ such that,

$$\|\alpha x^* + (1 - \alpha)y^*\|^2 \leq \alpha \|x^*\|^2 + (1 - \alpha)\|y^*\|^2 - \alpha(1 - \alpha)g(\|x^* - y^*\|),$$

for $x^*, y^* \in B_r^*$ and $\alpha \in [0, 1]$. So, for $p \in \Omega$ we have

$$\begin{aligned} \phi(p, u_n) &= \phi(p, \Theta_n^M y_n) \\ &\leq \phi(p, y_n) \\ &= \phi(p, j^{-1}(\alpha_n j x_n + (1 - \alpha_n) j A_N^{\lambda_n} x_n)) \\ &= \|p\|^2 - 2\langle p, \alpha_n j x_n + (1 - \alpha_n) j A_N^{\lambda_n} x_n \rangle + \|\alpha_n j x_n + (1 - \alpha_n) j A_N^{\lambda_n} x_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n \langle p, j x_n \rangle - 2(1 - \alpha_n) \langle p, j A_N^{\lambda_n} x_n \rangle + \alpha_n \|x_n\|^2 + (1 - \alpha_n) \|A_N^{\lambda_n} x_n\|^2 \\ &\quad - \alpha_n(1 - \alpha_n) g(\|j x_n - j A_N^{\lambda_n} x_n\|) \\ &= \alpha_n \phi(p, x_n) + (1 - \alpha_n) \phi(p, A_N^{\lambda_n} x_n) - \alpha_n(1 - \alpha_n) g(\|j x_n - j A_N^{\lambda_n} x_n\|) \\ &\leq \phi(p, x_n) - \alpha_n(1 - \alpha_n) g(\|j x_n - j A_N^{\lambda_n} x_n\|). \end{aligned}$$

Therefore, we have

$$\alpha_n(1 - \alpha_n) g(\|j x_n - j A_N^{\lambda_n} x_n\|) \leq \phi(p, x_n) - \phi(p, u_n), \quad \forall n \in \mathbb{N}.$$

Since

$$\begin{aligned} 0 &\leq \phi(p, x_n) - \phi(p, u_n) \\ &= \|x_n\|^2 - 2\langle p, j x_n - j u_n \rangle - \|u_n\|^2 \\ &\leq |\|x_n\|^2 - \|u_n\|^2| + 2|\langle p, j x_n - j u_n \rangle| \\ &\leq |\|x_n\| - \|u_n\|| (\|x_n\| + \|u_n\|) + 2\|p\| \|j x_n - j u_n\| \\ &\leq \|x_n - u_n\| (\|x_n\| + \|u_n\|) + 2\|p\| \|j x_n - j u_n\|, \end{aligned}$$

for $\{x_n\}, \{u_n\}$ being bounded and $\lim_{n \rightarrow \infty} \|j x_n - j u_n\| = \lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$, we have

$$\lim_{n \rightarrow \infty} \phi(p, x_n) - \phi(p, u_n) = 0.$$

From $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$, we have

$$\lim_{n \rightarrow \infty} g(\|j x_n - j A_N^{\lambda_n} x_n\|) = 0.$$

From the property of g , we have

$$\lim_{n \rightarrow \infty} \|j x_n - j A_N^{\lambda_n} x_n\| = 0.$$

As J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|x_n - A_N^{\lambda_n} x_n\| = 0.$$

Since $\{x_n\}$ is bounded then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow \hat{x}$ and together with the fact of $A_N^{\lambda_n} \in V$ we have $\hat{x} \in F(A_N^{\lambda_n}) \forall N \in \mathbb{N}$. But, $F(A_N^{\lambda_n}) = \cap_{i=1}^N F(S_i)$.

Next we show $\hat{x} \in EP(f_k) \forall k \in \{1, 2, \dots, M\}$. Now, from $u_n = \Theta_n^M y_n$, lemma 2.5 and $\phi(p, y_n) \leq \phi(p, x_n)$ we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(\Theta_n^M y_n, y_n) \\ &\leq \phi(p, y_n) - \phi(p, \Theta_n^M y_n) \\ &\leq \phi(p, x_n) - \phi(p, \Theta_n^M y_n) \\ &= \phi(p, x_n) - \phi(p, u_n). \end{aligned}$$

So, we have

$$\lim_{n \rightarrow \infty} \phi(u_n, y_n) = 0.$$

Since E is uniformly convex and smooth, we have from lemma 2.2 that

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \quad (9)$$

Using $\|x_n - u_n\| \rightarrow 0$ and (9) we have $y_{nk} \rightarrow \hat{x}$ and $u_{nk} \rightarrow \hat{x}$. For J norm-to-norm uniformly continuous on bounded sets and (9) we have

$$\lim_{n \rightarrow \infty} \|ju_n - jy_n\| = 0.$$

From $r_{k,n} \in (0, \infty) \forall k \in \{1, 2, \dots, M\}$ with $n, M \in \mathbb{N}$ we have

$$\lim_{n \rightarrow \infty} \frac{\|ju_n - jy_n\|}{r_{M,n}} = 0. \tag{10}$$

From $u_n = \Theta_n^M y_n$, we have

$$f_M(u_n, y) + \frac{1}{r_{M,n}} \langle y - u_n, ju_n - jy_n \rangle \geq 0, \quad \forall y \in C,$$

from which we have

$$f_M(u_{nj}, y) + \frac{1}{r_{M,nj}} \langle y - u_{nj}, ju_{nj} - jy_{nj} \rangle \geq 0,$$

and from (A2) we have

$$\begin{aligned} \frac{1}{r_{M,nj}} \langle y - u_{nj}, ju_{nj} - jy_{nj} \rangle &\geq -f_M(u_{nj}, y) \\ &\geq f_M(y, u_{nj}). \end{aligned}$$

Letting $j \rightarrow \infty$, we have from (10) and (A4) that

$$f_M(y, \hat{x}) \leq 0, \quad \forall y \in C.$$

Since, $f_k(x, y) \leq f_{k+1}(x, y) \forall x, y \in E$ then

$$f_k(y, \hat{x}) \leq 0, \quad \forall y \in C \text{ and } k \in \{1, 2, \dots, M\}.$$

For $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$ we have $y_t \in C$ and $f_k(y_t, \hat{x}) \leq 0 \forall k \in \{1, 2, \dots, M\}$. So, from (A1) and (A4) we have

$$\begin{aligned} 0 &= f_k(y_t, y_t) \\ &\leq t f_k(y_t, y) + (1 - t) f_k(y_t, \hat{x}) \\ &\leq t f_k(y_t, y), \quad \forall k \in \{1, 2, \dots, M\}. \end{aligned}$$

Dividing by t and letting $t \rightarrow 0$, together with (A3) we have

$$f_k(\hat{x}, y) \geq 0, \quad \forall y \in C \text{ and } \forall k \in \{1, 2, \dots, M\}.$$

Therefore, $\hat{x} \in EP(f_k) \forall k \in \{1, 2, \dots, M\}$, which implies that $\hat{x} \in \bigcap_{k=1}^M EP(f_k)$. Hence, $\hat{x} \in \Omega$.

Let $h = \Pi_{\Omega} x_1$ from $x_{n+1} = \Pi_{C_n \cap Q_n} x_1$ and $h \in \Omega \subset C_n \cap Q_n$, we have

$$\phi(x_{n+1}, x_1) \leq \phi(h, x_1).$$

Since a norm is weakly lower semi-continuous, we have

$$\begin{aligned} \phi(\hat{x}, x_1) &= \|\hat{x}\|^2 - 2\langle \hat{x}, jx_1 \rangle + \|x_1\|^2 \\ &\leq \liminf_{j \rightarrow \infty} (\|x_{nj}\|^2 - 2\langle x_{nj}, jx_1 \rangle + \|x_1\|^2) \\ &= \liminf_{j \rightarrow \infty} \phi(x_{nj}, x_1) \\ &\leq \limsup_{j \rightarrow \infty} \phi(x_{nj}, x_1) \\ &\leq \phi(h, x_1). \end{aligned}$$

From the definition of Π_{Ω} , we have $\hat{x} = h$. Hence, $\lim_{j \rightarrow \infty} \phi(x_{nj}, x_1) = \phi(h, x_1)$ therefore,

$$\begin{aligned} 0 &= \lim_{j \rightarrow \infty} \phi(x_{nj}, x_1) - \phi(h, x_1) \\ &= \lim_{j \rightarrow \infty} (\|x_{nj}\|^2 - \|h\|^2 - 2\langle x_{nj} - h, Jx_1 \rangle) \\ &= \lim_{j \rightarrow \infty} (\|x_{nj}\|^2 - \|h\|^2). \end{aligned}$$

Since E has the Kadec-Klee property, we have that $x_{nj} \rightarrow h = \Pi_{\Omega}x_1$, hence $\{x_n\}$ converges strongly to $\Pi_{\Omega}x_1$. \square

4 Conclusion

Nonlinear operators are of vital importance in the study/discussion of fixed point theory. It is clear that, the result obtained in this research improves the work in [15] by using a linear combination of relatively nonexpansive mappings compared to the single relatively nonexpansive map used in [15]. Also, establishing a result with different classes of nonlinear operators rather than only one class is an achievement; as the application of the result can now have wider coverage. Although, the research conducted adopt almost all the procedures and technique used in [21] with some little adjustments where necessary but, the mappings used in [21] were substituted with relatively nonexpansive mappings due to their application importance especially in resolvent of a maximal monotone operators as can be seen in [22] and therein. Future research can be conducted by using a linear combination of both the relatively nonexpansive and ϕ -nonexpansive mappings in a single algorithm or just using mappings of more application importance.

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